

# 1

# Functions and Models



# 1.1

# Four Ways to Represent a Function

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# Four Ways to Represent a Function

Functions arise whenever one quantity depends on another. Consider the following four situations.

- A.** The area  $A$  of a circle depends on the radius  $r$  of the circle. The rule that connects  $r$  and  $A$  is given by the equation  $A = \pi r^2$ . With each positive number  $r$  there is associated one value of  $A$ , and we say that  $A$  is a *function* of  $r$ .

# Four Ways to Represent a Function

**B.** The human population of the world  $P$  depends on the time  $t$ . The table gives estimates of the world population  $P(t)$  at time  $t$ , for certain years. For instance,

$$P(1950) \approx 2,560,000,000$$

But for each value of the time  $t$  there is a corresponding value of  $P$ , and we say that  $P$  is a function of  $t$ .

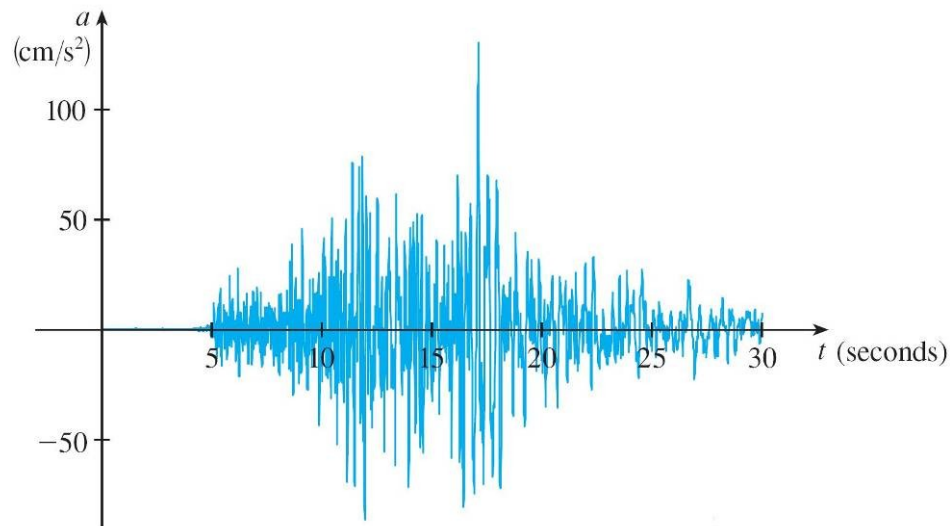
Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

# Four Ways to Represent a Function

- C.** The cost  $C$  of mailing a large envelope depends on the weight  $w$  of the envelope. Although there is no simple formula that connects  $w$  and  $C$ , the post office has a rule for determining  $C$  when  $w$  is known.
- D.** The vertical acceleration  $a$  of the ground as measured by a seismograph during an earthquake is a function of the elapsed time  $t$ .

# Four Ways to Represent a Function

Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of  $t$ , the graph provides a corresponding value of  $a$ .



Vertical ground acceleration during the Northridge earthquake

Figure 1

# Four Ways to Represent a Function

A **function**  $f$  is a rule that assigns to each element  $x$  in a set  $D$  exactly one element, called  $f(x)$ , in a set  $E$ .

We usually consider functions for which the sets  $D$  and  $E$  are sets of real numbers. The set  $D$  is called the **domain** of the function.

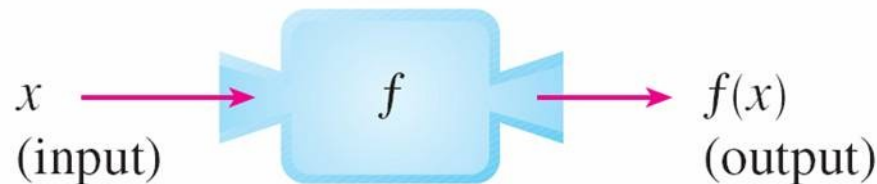
The number  $f(x)$  is the **value of  $f$  at  $x$**  and is read “ $f$  of  $x$ .” The **range** of  $f$  is the set of all possible values of  $f(x)$  as  $x$  varies throughout the domain.

A symbol that represents an arbitrary number in the *domain* of a function  $f$  is called an **independent variable**.

# Four Ways to Represent a Function

A symbol that represents a number in the *range* of  $f$  is called a **dependent variable**. In Example A, for instance,  $r$  is the independent variable and  $A$  is the dependent variable.

It's helpful to think of a function as a **machine** (see Figure 2).



Machine diagram for a function

Figure 2

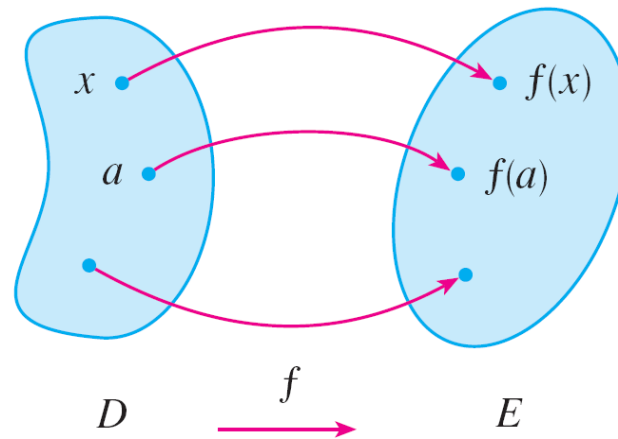
# Four Ways to Represent a Function

If  $x$  is in the domain of the function  $f$ , then when  $x$  enters the machine, it's accepted as an input and the machine produces an output  $f(x)$  according to the rule of the function.

Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

# Four Ways to Represent a Function

Another way to picture a function is by an **arrow diagram** as in Figure 3.



Arrow diagram for  
Figure 3

Each arrow connects an element of  $D$  to an element of  $E$ . The arrow indicates that  $f(x)$  is associated with  $x$ ,  $f(a)$  is associated with  $a$ , and so on.

# Four Ways to Represent a Function

The most common method for visualizing a function is its graph. If  $f$  is a function with domain  $D$ , then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

In other words, the graph of  $f$  consists of all points  $(x, y)$  in the coordinate plane such that  $y = f(x)$  and  $x$  is in the domain of  $f$ .

The graph of a function  $f$  gives us a useful picture of the behavior or “life history” of a function.

# Four Ways to Represent a Function

Since the  $y$ -coordinate of any point  $(x, y)$  on the graph is  $y = f(x)$ , we can read the value of  $f(x)$  from the graph as being the height of the graph above the point  $x$  (see Figure 4).

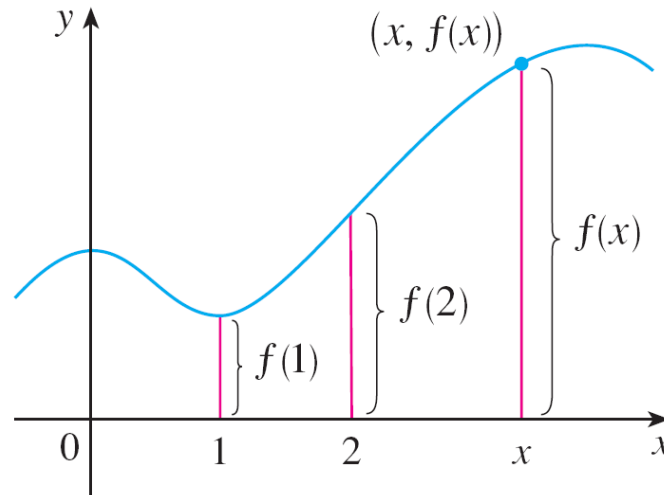


Figure 4

# Four Ways to Represent a Function

The graph of  $f$  also allows us to picture the domain of  $f$  on the  $x$ -axis and its range on the  $y$ -axis as in Figure 5.

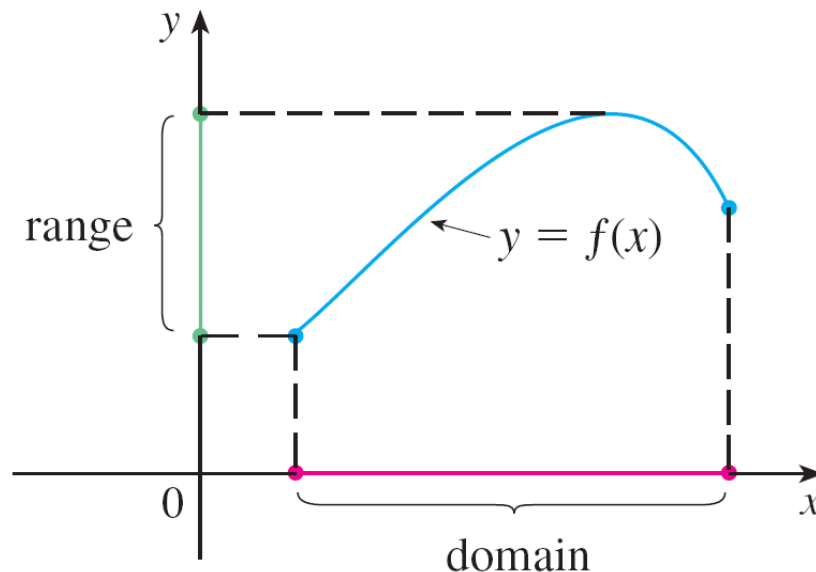


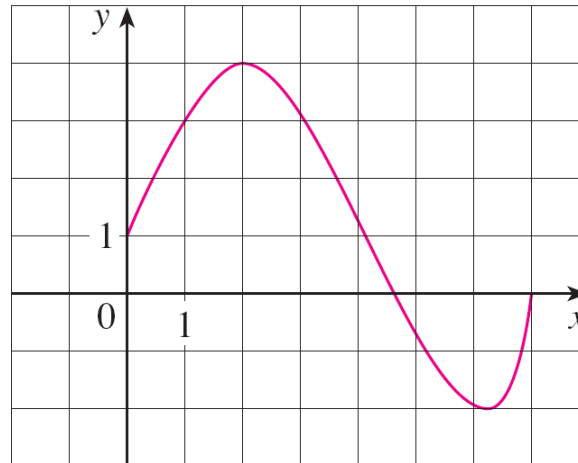
Figure 5

# Example 1

The graph of a function  $f$  is shown in Figure 6.

**(a)** Find the values of  $f(1)$  and  $f(5)$ .

**(b)** What are the domain and range of  $f$ ?



**Figure 6**

The notation for intervals is given in Appendix A.

# Example 1 – Solution

**(a)** We see from Figure 6 that the point  $(1, 3)$  lies on the graph of  $f$ , so the value of  $f$  at 1 is  $f(1) = 3$ . (In other words, the point on the graph that lies above  $x = 1$  is 3 units above the  $x$ -axis.)

When  $x = 5$ , the graph lies about 0.7 unit below the  $x$ -axis, so we estimate that  $f(5) \approx -0.7$ .

**(b)** We see that  $f(x)$  is defined when  $0 \leq x \leq 7$ , so the domain of  $f$  is the closed interval  $[0, 7]$ . Notice that  $f$  takes on all values from  $-2$  to  $4$ , so the range of  $f$  is

$$\{y \mid -2 \leq y \leq 4\} = [-2, 4]$$



# Representations of Functions

# Representations of Functions

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

# Example 4

When you turn on a hot-water faucet, the temperature  $T$  of the water depends on how long the water has been running. Draw a rough graph of  $T$  as a function of the time  $t$  that has elapsed since the faucet was turned on.

## Solution:

The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes.

When the water from the hot-water tank starts flowing from the faucet,  $T$  increases quickly. In the next phase,  $T$  is constant at the temperature of the heated water in the tank.

# Example 4 – *Solution*

cont'd

When the tank is drained,  $T$  decreases to the temperature of the water supply.

This enables us to make the rough sketch of  $T$  as a function of  $t$  in Figure 11.

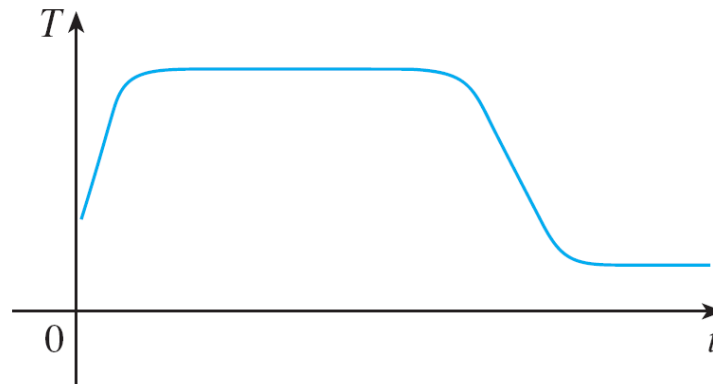


Figure 11

# Representations of Functions

The graph of a function is a curve in the  $xy$ -plane. But the question arises: Which curves in the  $xy$ -plane are graphs of functions? This is answered by the following test.

**The Vertical Line Test** A curve in the  $xy$ -plane is the graph of a function of  $x$  if and only if no vertical line intersects the curve more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 13.

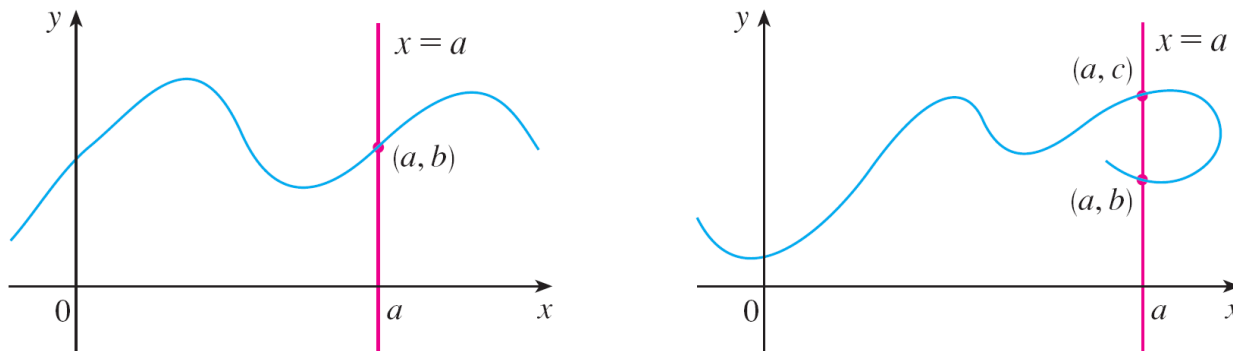


Figure 13

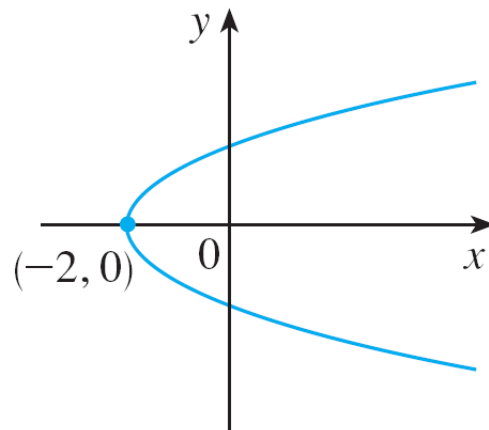
# Representations of Functions

If each vertical line  $x = a$  intersects a curve only once, at  $(a, b)$ , then exactly one functional value is defined by  $f(a) = b$ .

But if a line  $x = a$  intersects the curve twice, at  $(a, b)$  and  $(a, c)$ , then the curve can't represent a function because a function can't assign two different values to  $a$ .

# Representations of Functions

For example, the parabola  $x = y^2 - 2$  shown in Figure 14(a) is not the graph of a function of  $x$  because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of  $x$ .



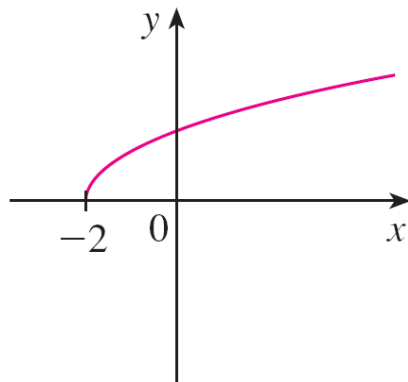
$$x = y^2 - 2$$

Figure 14(a)

# Representations of Functions

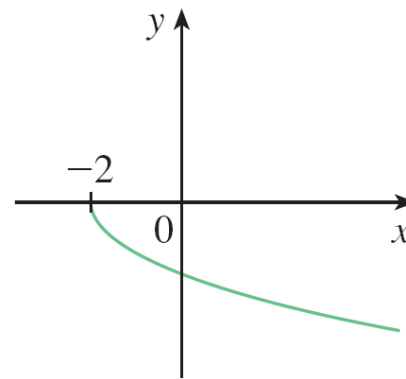
Notice that the equation  $x = y^2 - 2$  implies  $y^2 = x + 2$ , so  $y = \pm\sqrt{x + 2}$ .

Thus the upper and lower halves of the parabola are the graphs of the functions  $f(x) = \sqrt{x + 2}$  and  $g(x) = -\sqrt{x + 2}$   
[See Figures 14(b) and (c).]



$$y = \sqrt{x + 2}$$

Figure 14(b)



$$y = -\sqrt{x + 2}$$

Figure 14(c)

# Representations of Functions

We observe that if we reverse the roles of  $x$  and  $y$ , then the equation  $x = h(y) = y^2 - 2$  *does* define  $x$  as a function of  $y$  (with  $y$  as the independent variable and  $x$  as the dependent variable) and the parabola now appears as the graph of the function  $h$ .



# Piecewise Defined Functions

# Example 7

A function  $f$  is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate  $f(-2)$ ,  $f(-1)$ , and  $f(0)$  and sketch the graph.

**Solution:**

Remember that a function is a rule. For this particular function the rule is the following:

First look at the value of the input  $x$ . If it happens that  $x \leq -1$ , then the value of  $f(x)$  is  $1 - x$ .

# Example 7 – Solution

cont'd

On the other hand, if  $x > -1$ , then the value of  $f(x)$  is  $x^2$ .

Since  $-2 \leq -1$ , we have  $f(-2) = 1 - (-2) = 3$ .

Since  $-1 \leq -1$ , we have  $f(-1) = 1 - (-1) = 2$ .

Since  $0 > -1$ , we have  $f(0) = 0^2 = 0$ .

How do we draw the graph of  $f$ ? We observe that if  $x \leq -1$ , then  $f(x) = 1 - x$ , so the part of the graph of  $f$  that lies to the left of the vertical line  $x = -1$  must coincide with the line  $y = 1 - x$ , which has slope  $-1$  and  $y$ -intercept  $1$ .

# Example 7 – Solution

cont'd

If  $x > -1$ , then  $f(x) = x^2$ , so the part of the graph of  $f$  that lies to the right of the line  $x = -1$  must coincide with the graph of  $y = x^2$ , which is a parabola. This enables us to sketch the graph in Figure 15.

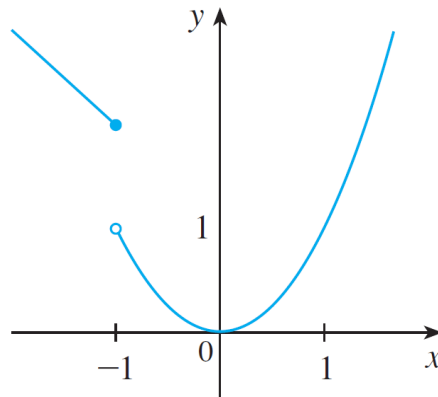


Figure 15

The solid dot indicates that the point  $(-1, 2)$  is included on the graph; the open dot indicates that the point  $(-1, 1)$  is excluded from the graph.

# Piecewise Defined Functions

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number  $a$ , denoted by  $|a|$ , is the distance from  $a$  to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geq 0 \quad \text{for every number } a$$

For example,

$$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1$$

$$|3 - \pi| = \pi - 3$$

# Piecewise Defined Functions

In general, we have

$$|a| = a \quad \text{if } a \geq 0$$

$$|a| = -a \quad \text{if } a < 0$$

(Remember that if  $a$  is negative, then  $-a$  is positive.)

# Example 8

Sketch the graph of the absolute value function  $f(x) = |x|$ .

**Solution:**

From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

# Example 8 – Solution

cont'd

Using the same method as in Example 7, we see that the graph of  $f$  coincides with the line  $y = x$  to the right of the  $y$ -axis and coincides with the line  $y = -x$  to the left of the  $y$ -axis (see Figure 16).

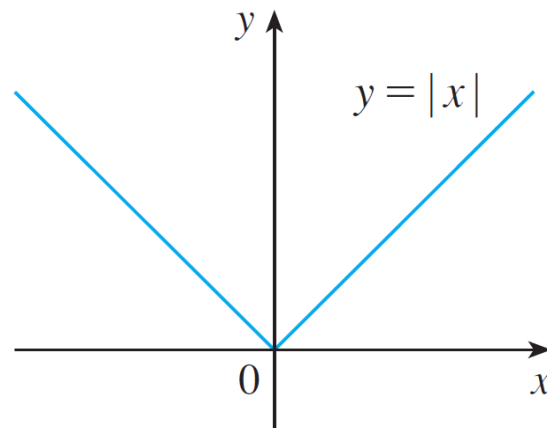


Figure 16

# Example 10

In Example C at the beginning of this section we considered the cost  $C(w)$  of mailing a large envelope with weight  $w$ .

In effect, this is a piecewise defined function because, from the table of values on page 13, we have

$$C(w) = \begin{cases} 0.88 & \text{if } 0 < w \leq 1 \\ 1.05 & \text{if } 1 < w \leq 2 \\ 1.22 & \text{if } 2 < w \leq 3 \\ 1.39 & \text{if } 3 < w \leq 4 \\ \vdots & \\ \vdots & \\ \vdots & \end{cases}$$

# Example 10

cont'd

The graph is shown in Figure 18.

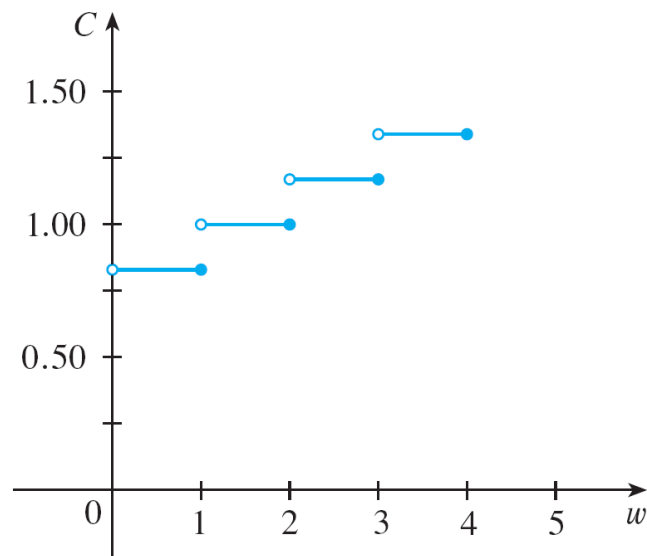


Figure 18

You can see why functions similar to this one are called **step functions**—they jump from one value to the next.



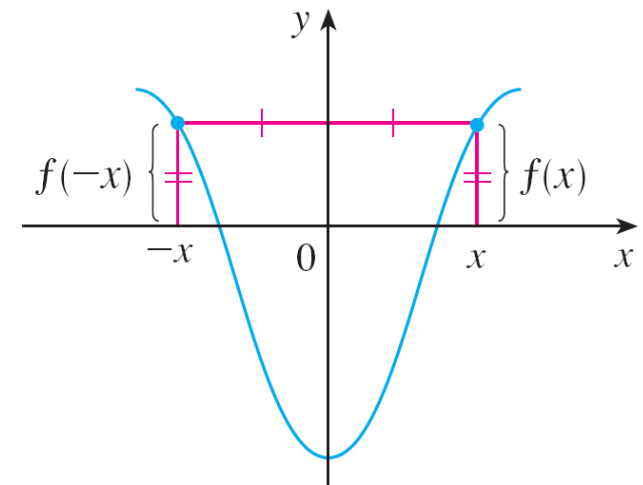
# Symmetry

# Symmetry

If a function  $f$  satisfies  $f(-x) = f(x)$  for every number  $x$  in its domain, then  $f$  is called an **even function**. For instance, the function  $f(x) = x^2$  is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the  $y$ -axis (see Figure 19).



An even  
function  
Figure 19

# Symmetry

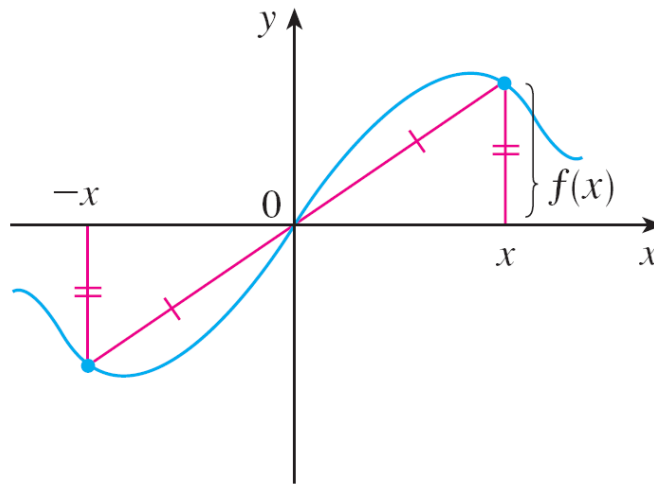
This means that if we have plotted the graph of  $f$  for  $x \geq 0$ , we obtain the entire graph simply by reflecting this portion about the  $y$ -axis.

If  $f$  satisfies  $f(-x) = -f(x)$  for every number  $x$  in its domain, then  $f$  is called an **odd function**. For example, the function  $f(x) = x^3$  is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

# Symmetry

The graph of an odd function is symmetric about the origin (see Figure 20).



An odd  
function  
figure 20

If we already have the graph of  $f$  for  $x \geq 0$ , we can obtain the entire graph by rotating this portion through  $180^\circ$  about the origin.

# Example 11

Determine whether each of the following functions is even, odd, or neither even nor odd.

$$(a) f(x) = x^5 + x \quad (b) g(x) = 1 - x^4 \quad (c) h(x) = 2x - x^2$$

**Solution:**

$$(a) f(-x) = (-x)^5 + (-x) = (-1)^5 x^5 + (-x)$$

$$= -x^5 - x = -(x^5 + x)$$

$$= -f(x)$$

Therefore  $f$  is an odd function.

# Example 11 – Solution

cont'd

$$(b) \ g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So  $g$  is even.

$$(c) \ h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since  $h(-x) \neq h(x)$  and  $h(-x) \neq -h(x)$ , we conclude that  $h$  is neither even nor odd.

# Symmetry

The graphs of the functions in Example 11 are shown in Figure 21. Notice that the graph of  $h$  is symmetric neither about the  $y$ -axis nor about the origin.

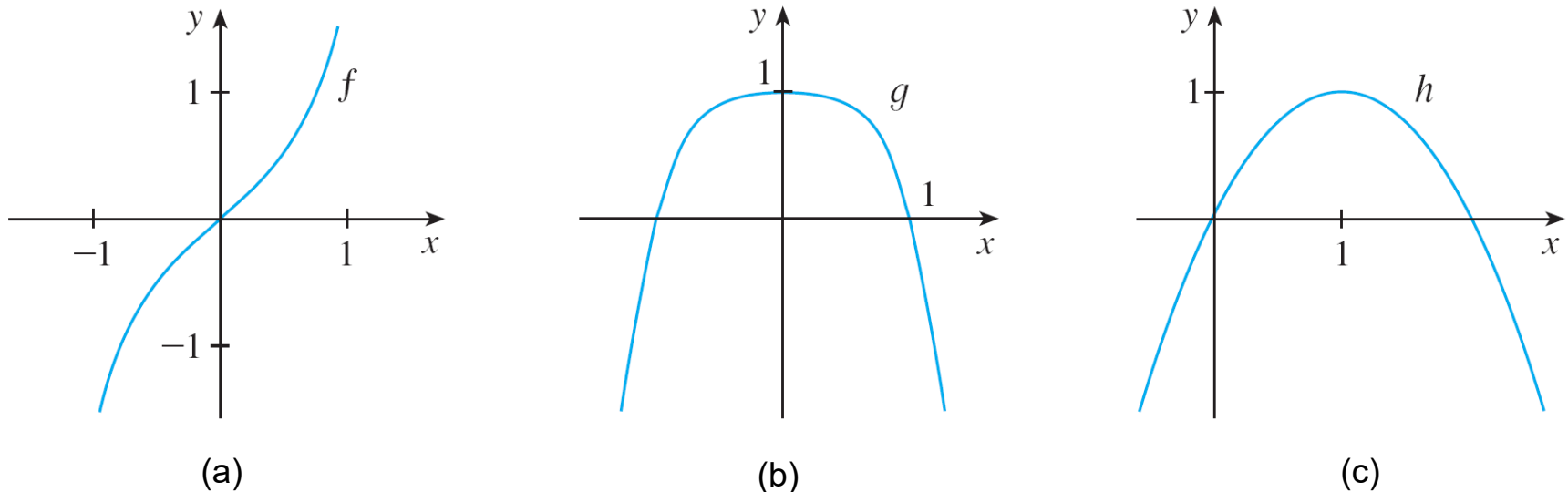


Figure 21



# Increasing and Decreasing Functions

# Increasing and Decreasing Functions

The graph shown in Figure 22 rises from  $A$  to  $B$ , falls from  $B$  to  $C$ , and rises again from  $C$  to  $D$ . The function  $f$  is said to be increasing on the interval  $[a, b]$ , decreasing on  $[b, c]$ , and increasing again on  $[c, d]$ .

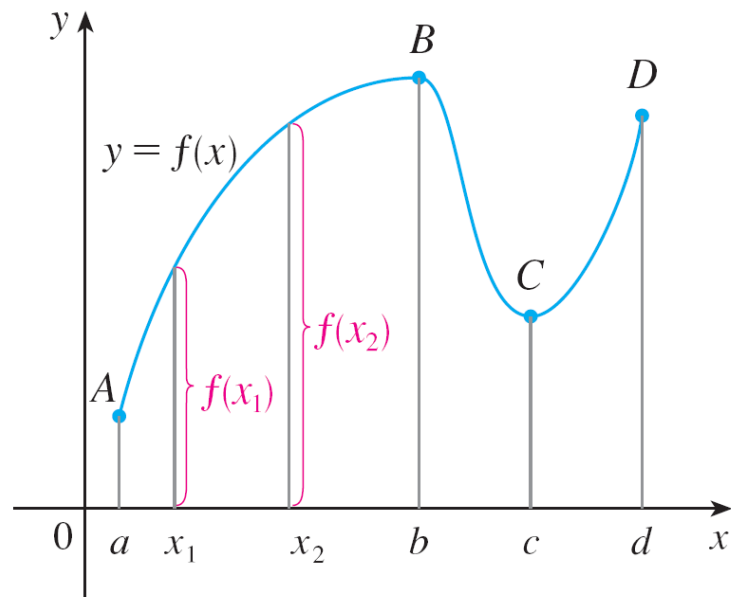


Figure 22

# Increasing and Decreasing Functions

Notice that if  $x_1$  and  $x_2$  are any two numbers between  $a$  and  $b$  with  $x_1 < x_2$ , then  $f(x_1) < f(x_2)$ .

We use this as the defining property of an increasing function.

A function  $f$  is called **increasing** on an interval  $I$  if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on  $I$  if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

# Increasing and Decreasing Functions

In the definition of an increasing function it is important to realize that the inequality  $f(x_1) < f(x_2)$  must be satisfied for every pair of numbers  $x_1$  and  $x_2$  in  $I$  with  $x_1 < x_2$ .

You can see from Figure 23 that the function  $f(x) = x^2$  is decreasing on the interval  $(-\infty, 0]$  and increasing on the interval  $[0, \infty)$ .

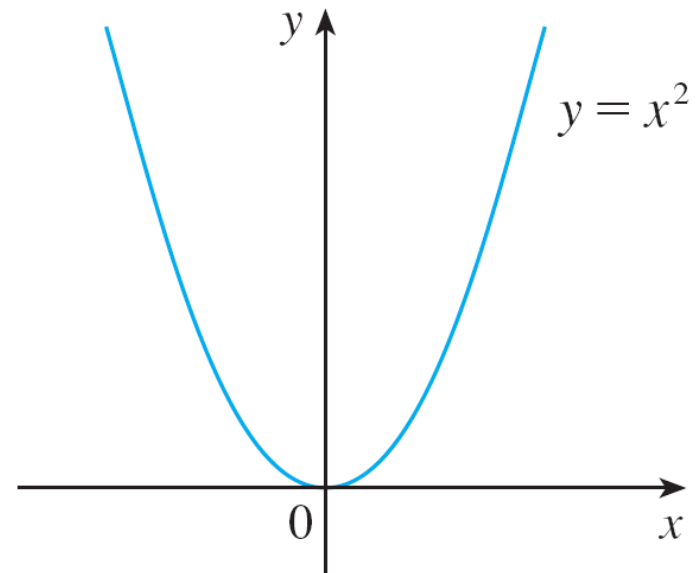


Figure 23