# 1 Functions and Models

# 1.2

# Mathematical Models: A Catalog of Essential Functions

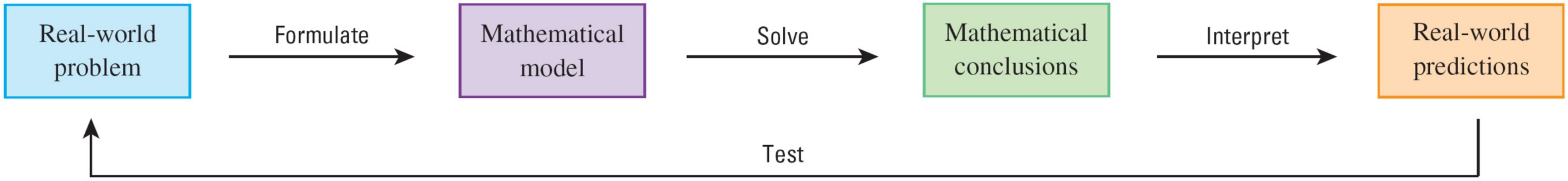
# Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions.

The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

# Mathematical Models: A Catalog of Essential Functions

Figure 1 illustrates the process of mathematical modeling.




The modeling process

**Figure 1**

# Mathematical Models: A Catalog of Essential Functions

A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization.* A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions.

It is important to realize the limitations of the model. In the end, Mother Nature has the final say.

There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

# Linear Models

# Linear Models

When we say that $y$ is a **linear function** of $x$, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

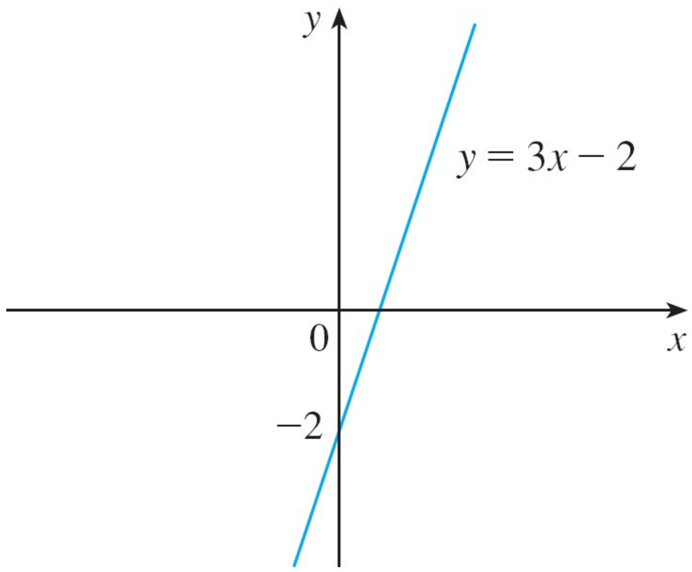
$$y = f(x) = mx + b$$

where $m$ is the slope of the line and $b$ is the $y$-intercept.

# Linear Models

A characteristic feature of linear functions is that they grow at a constant rate.

For instance, Figure 2 shows a graph of the linear function $f(x) = 3x - 2$ and a table of sample values.




| $x$ | $f(x) = 3x - 2$ |
|---|---|
| 1.0 | 1.0 |
| 1.1 | 1.3 |
| 1.2 | 1.6 |
| 1.3 | 1.9 |
| 1.4 | 2.2 |
| 1.5 | 2.5 |

**Figure 2**

# Linear Models

Notice that whenever $x$ increases by 0.1, the value of $f(x)$ increases by 0.3.

So $f(x)$ increases three times as fast as $x$. Thus the slope of the graph $y = 3x – 2$, namely 3, can be interpreted as the rate of change of $y$ with respect to $x$.

# Example 1

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature

$T$ (in °C) as a function of the height $h$ (in kilometers), assuming that a linear model is appropriate.

(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of 2.5 km?

# Example 1(a) – *Solution*

Because we are assuming that $T$ is a linear function of $h$, we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \cdot 0 + b = b$$

In other words, the $y$-intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$, so

$$10 = m \cdot 1 + 20$$

The slope of the line is therefore $m = 10 - 20 = -10$ and the required linear function is

$$T = -10h + 20$$

# Example 1(b) – *Solution*

cont'd

The graph is sketched in Figure 3.

The slope is $m = –10°C/km$, and this represents the rate of change of temperature with respect to height.

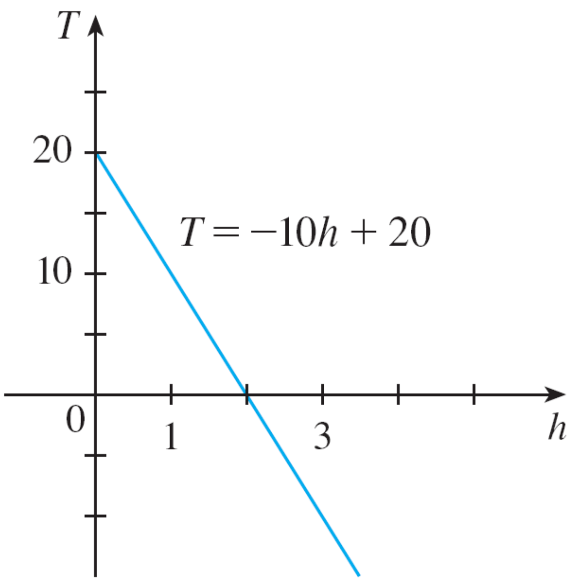



**Figure 3**

# Example 1(c) – *Solution*

cont'd

At a height of $h = 2.5$ km, the temperature is

$$T = -10(2.5) + 20 = -5°\text{C}$$

# Linear Models

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data.

We seek a curve that "fits" the data in the sense that it captures the basic trend of the data points.

# Polynomials

# Polynomials

A function $P$ is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $n$ is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial.

The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is $n$. For example, the function

$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

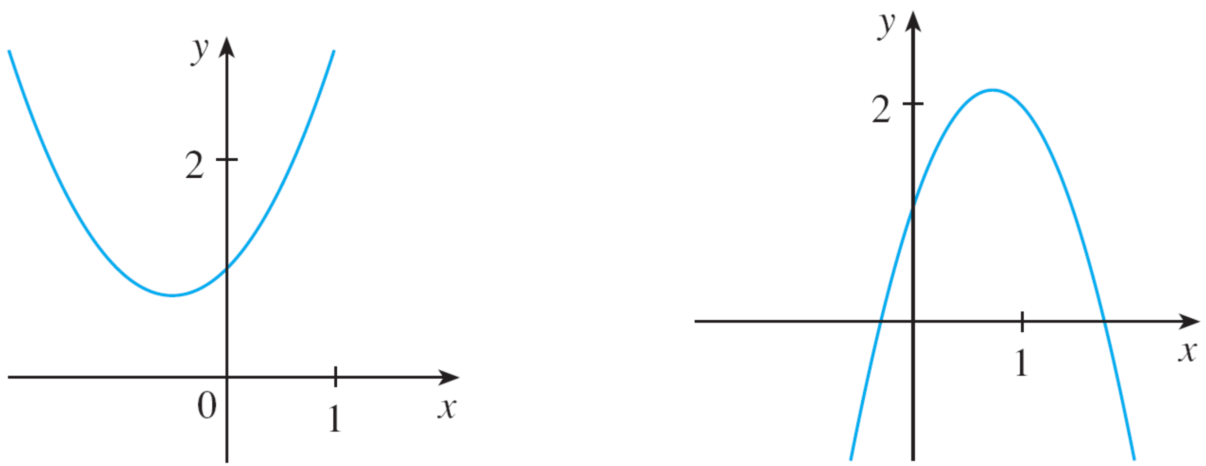
# Polynomials

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function.

A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**.

# Polynomials

Its graph is always a parabola obtained by shifting the parabola $y = ax^2$. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 7.)




(a) $y = x^2 + x + 1$ (b) $y = -2x^2 + 3x + 1$

The graphs of quadratic functions are parabolas.

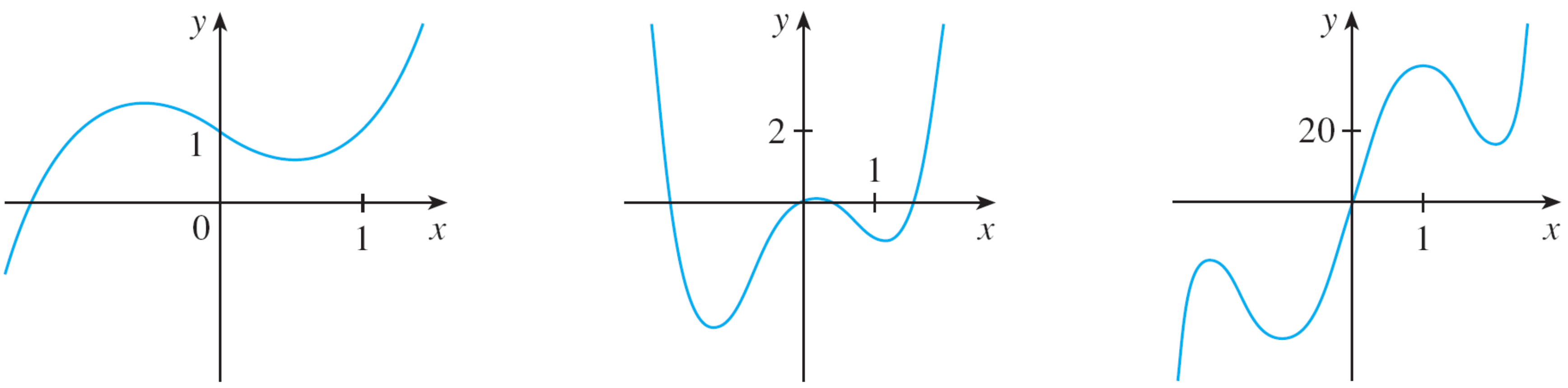
**Figure 7**

# Polynomials

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \qquad a \neq 0$$

and is called a **cubic function**. Figure 8 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c).




(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

**Figure 8**

# Example 4

A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height *h* above the ground is recorded at 1-second intervals in Table 2.

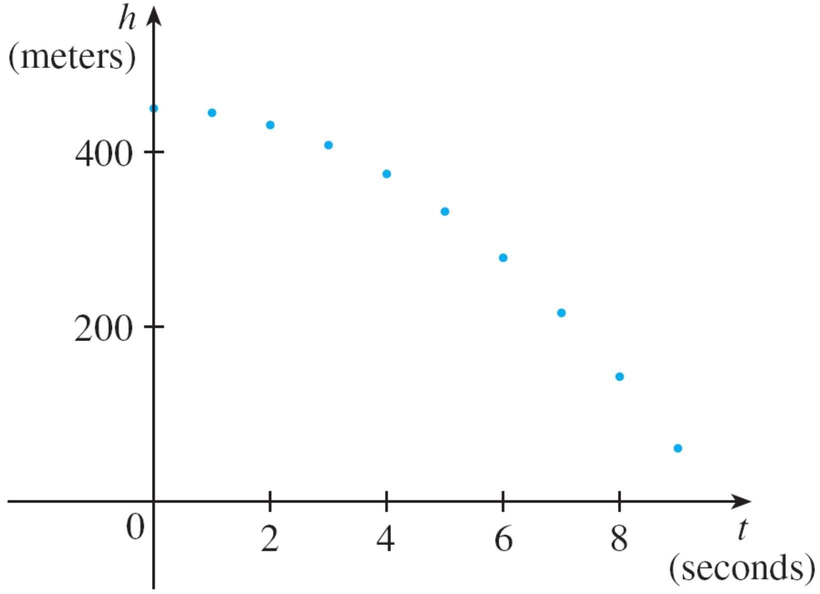
Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

**TABLE 2**

| Time (seconds) | Height (meters) |
|---|---|
| 0 | 450 |
| 1 | 445 |
| 2 | 431 |
| 3 | 408 |
| 4 | 375 |
| 5 | 332 |
| 6 | 279 |
| 7 | 216 |
| 8 | 143 |
| 9 | 61 |

# Example 4 – *Solution*

We draw a scatter plot of the data in Figure 9 and observe that a linear model is inappropriate.




Scatter plot for a falling ball

**Figure 9**

# Example 4 – *Solution*

cont'd

But it looks as if the data points might lie on a parabola, so we try a quadratic model instead.

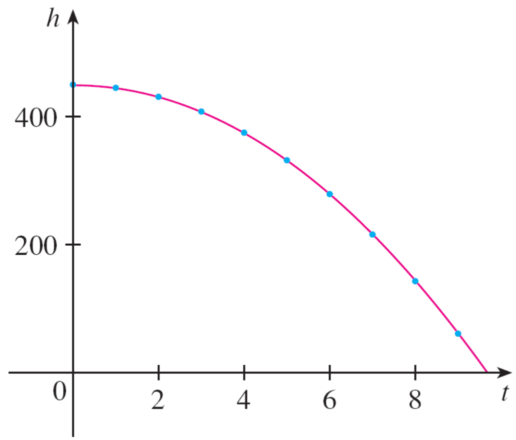
Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

**3** $$h = 449.36 + 0.96t - 4.90t^2$$

# Example 4 – *Solution*

cont'd

In Figure 10 we plot the graph of Equation 3 together with the data points and see that the quadratic model gives a very good fit.




Quadratic model for a falling ball

**Figure 10**

The ball hits the ground when $h = 0$, so we solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0$$

# Example 4 – *Solution*

cont'd

The quadratic formula gives

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

The positive root is $t \approx 9.67$, so we predict that the ball will hit the ground after about 9.7 seconds.

# Power Functions

# Power Functions

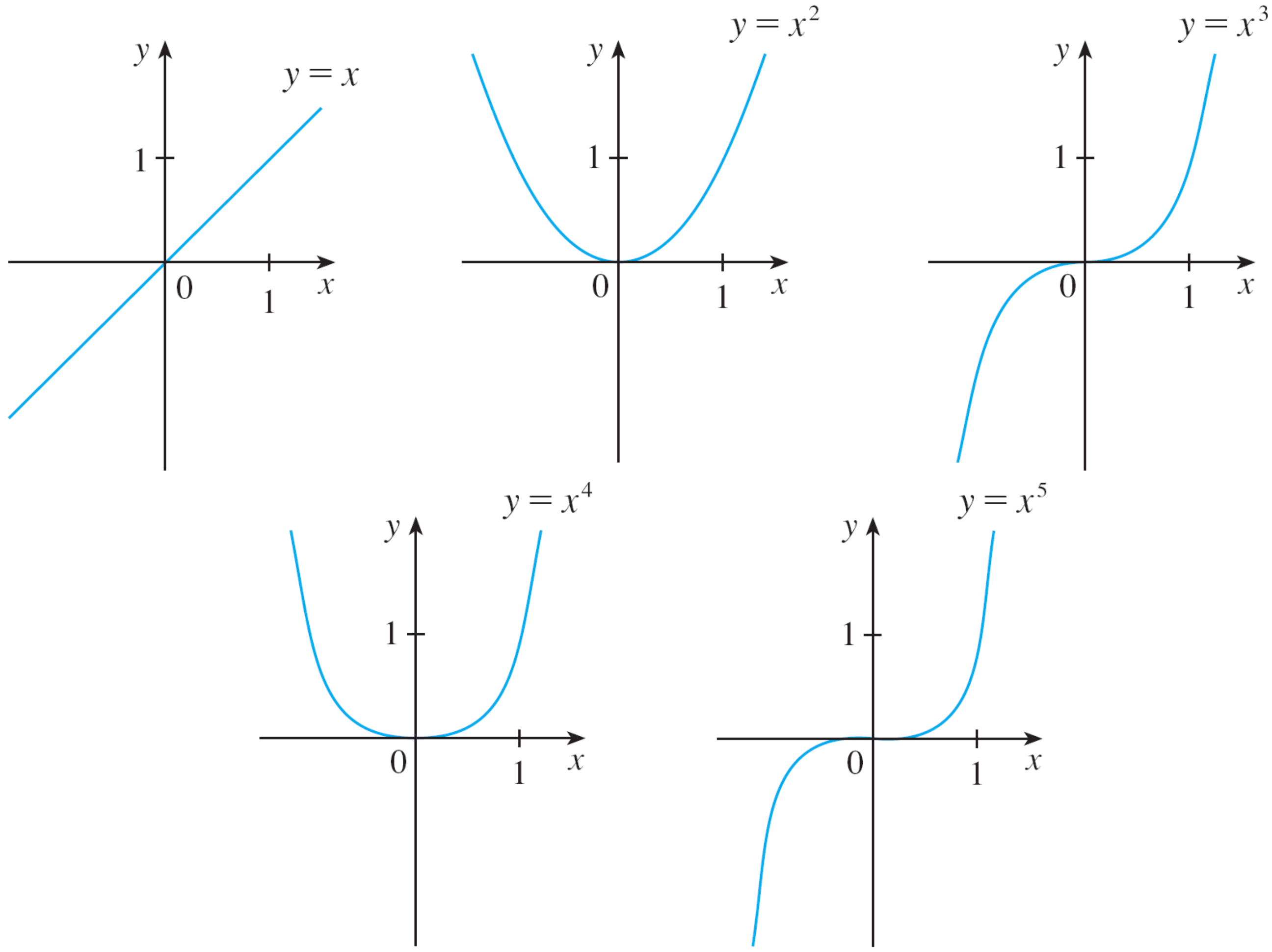
A function of the form $f(x) = x^a$, where $a$ is a constant, is called a **power function**. We consider several cases.

**(i) *a* = *n*, where *n* is a positive integer**

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 11. (These are polynomials with only one term.)

We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ (a parabola).

# Power Functions




Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

**Figure 11**

# Power Functions

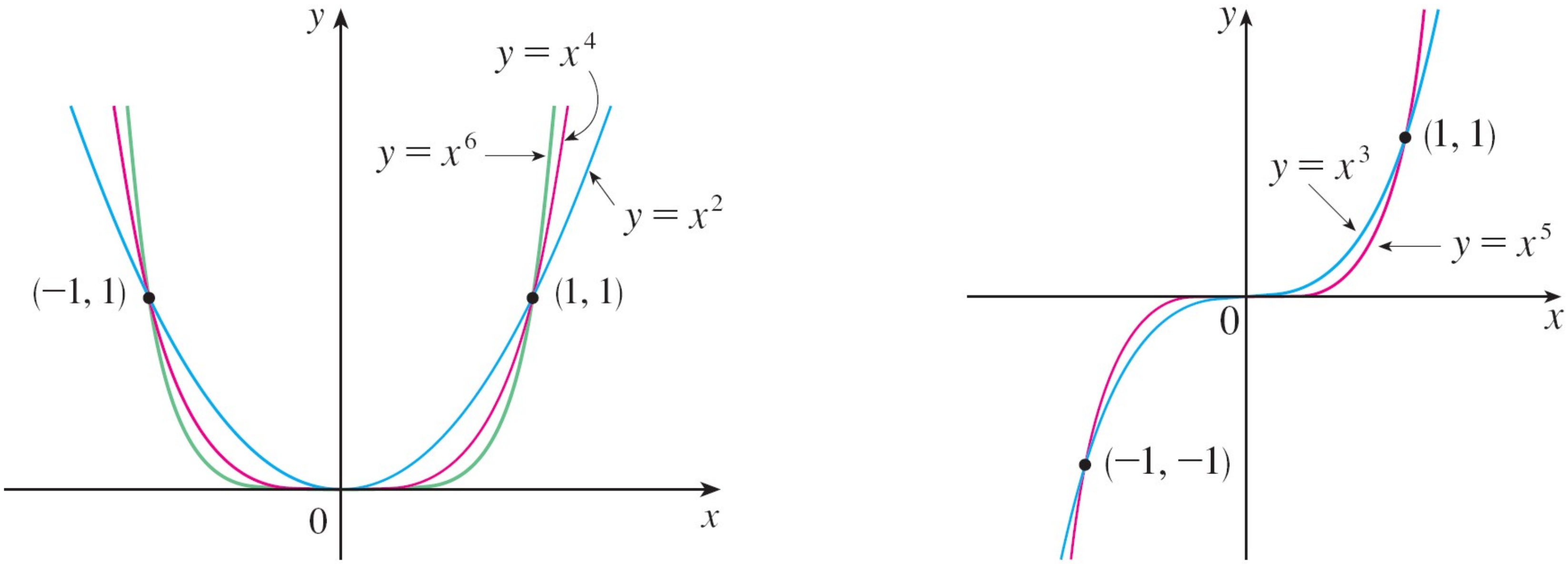
The general shape of the graph of $f(x) = x^n$ depends on whether $n$ is even or odd.

If $n$ is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$.

If $n$ is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$.

# Power Functions

Notice from Figure 12, however, that as $n$ increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$. (If $x$ is small, then $x^2$ is smaller, $x^3$ is even smaller, $x^4$ is smaller still, and so on.)



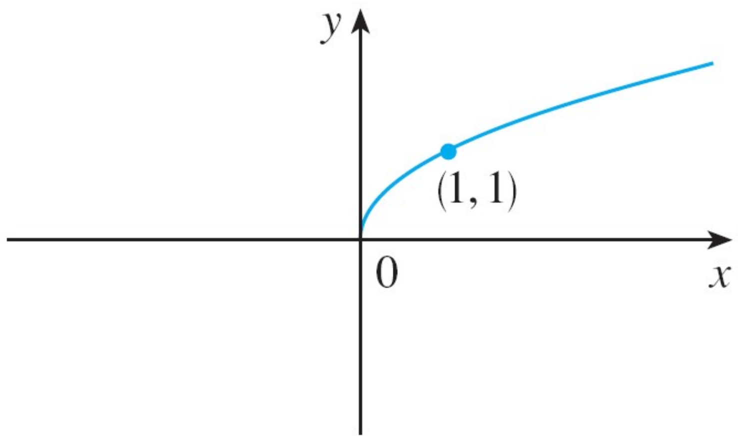

Families of power functions

**Figure 12**

# Power Functions

**(ii) *a* = 1/*n*, where *n* is a positive integer**

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For $n$ = 2 it is the square root functio $f(x) = \sqrt{x}$, whose domain is [0, ∞) and whose graph is the upper half of the parabola $x = y^2$. [See Figure 13(a).]




$f(x) = \sqrt{x}$

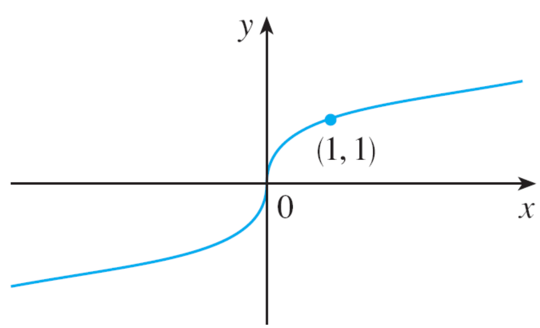
Graph of root function

**Figure 13(a)**

# Power Functions

For other even values of *n*, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$.

For *n* = 3 we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown in Figure 13(b). The graph of $y = \sqrt[n]{x}$ for *n* odd … that of $y = \sqrt[3]{x}$.




$f(x) = \sqrt[3]{x}$

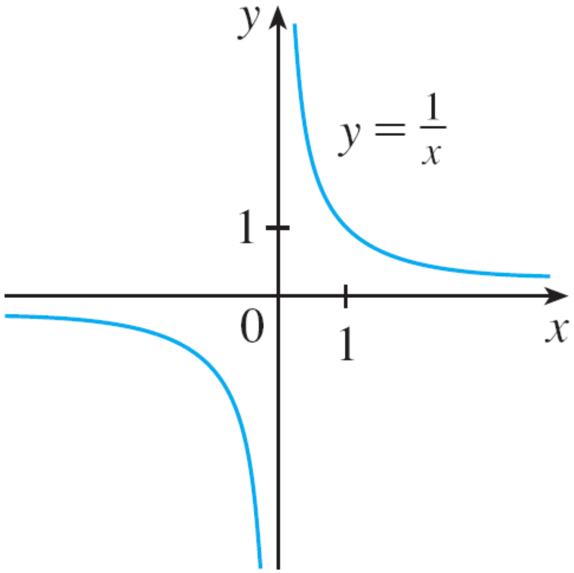
Graph of root function

**Figure 13(b)**

# Power Functions

**(iii)** ***a* = –1**

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 14. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes.




The reciprocal function

**Figure 14**

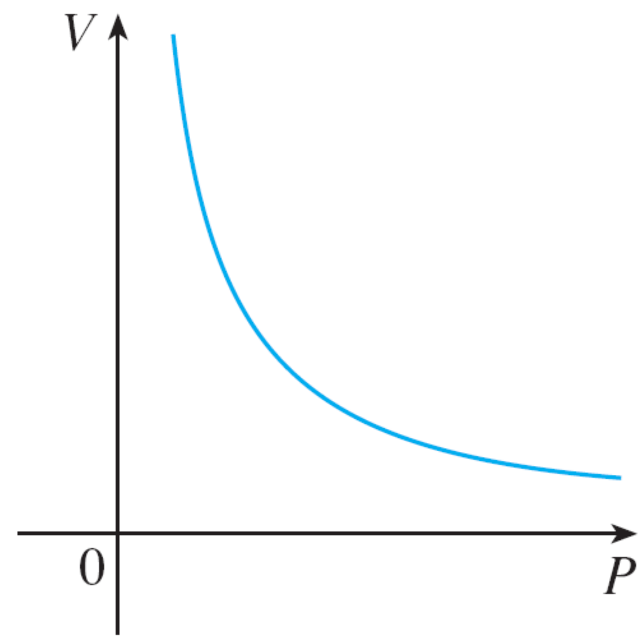
# Power Functions

This function arises in physics and chemistry in connection with Boyle's Law, which says that, when the temperature is constant, the volume *V* of a gas is inversely proportional to the pressure *P*:

$$V = \frac{C}{P}$$

where *C* is a constant.

Thus the graph of *V* as a function of *P* (see Figure 15) has the same general shape as the right half of Figure 14.




Volume as a function of pressure at constant temperature

**Figure 15**

# Rational Functions

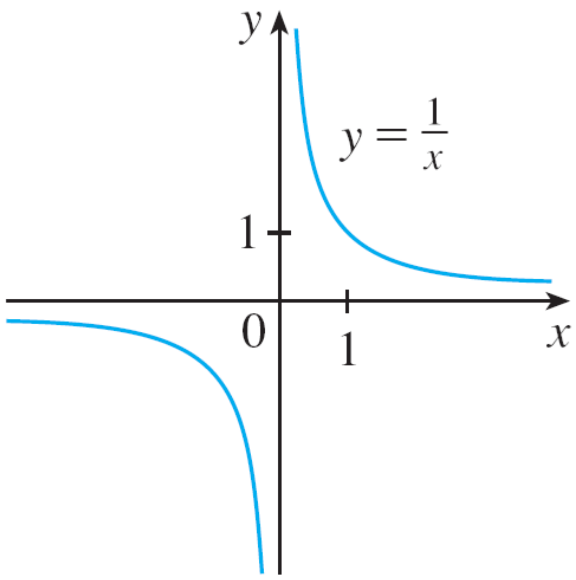
# Rational Functions

A **rational function** *f* is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where *P* and *Q* are polynomials. The domain consists of all values of *x* such that $Q(x) \neq 0$.

A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14.



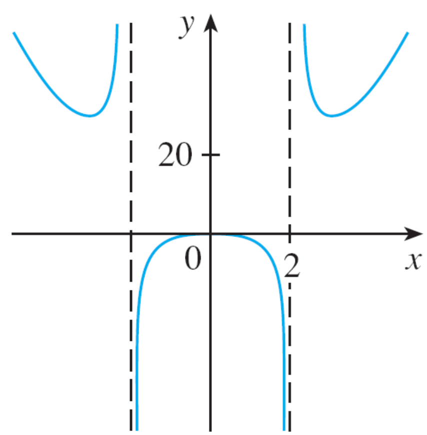

The reciprocal function
**Figure 14**

# Rational Functions

The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 16.




$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

**Figure 16**

# Algebraic Functions

# Algebraic Functions

A function *f* is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function.

Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

# Algebraic Functions

The graphs of algebraic functions can assume a variety of shapes. Figure 17 illustrates some of the possibilities.

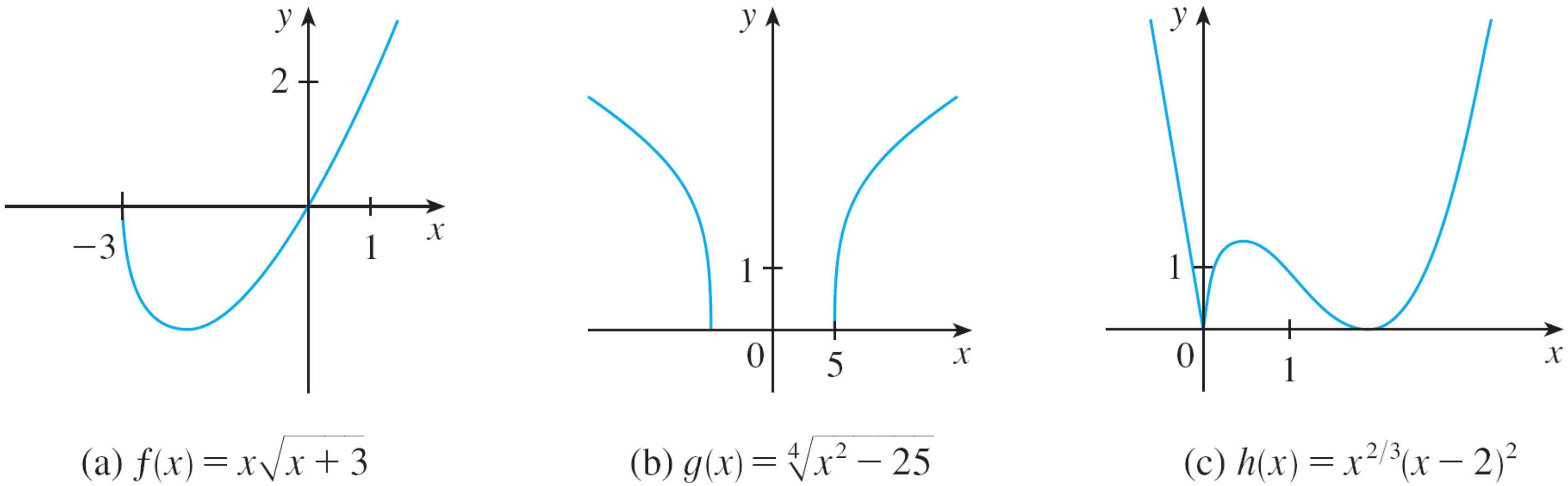



**Figure 17**

# Algebraic Functions

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity *v* is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where $m_0$ is the rest mass of the particle and $c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

# Trigonometric Functions

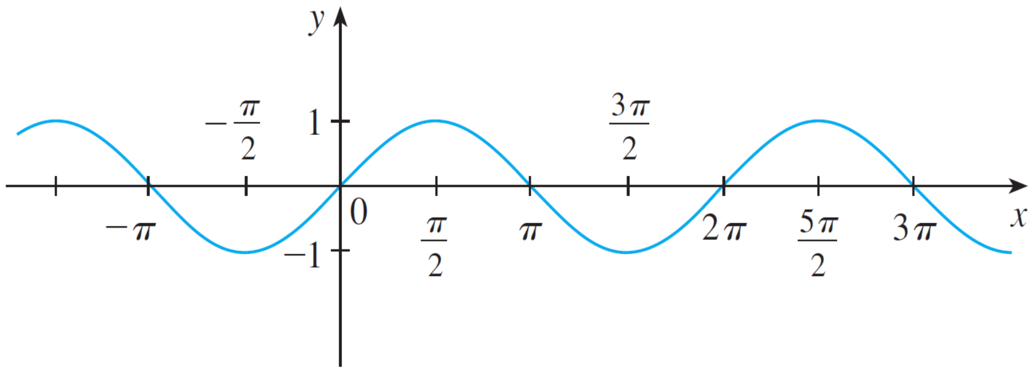
# Trigonometric Functions

In calculus the convention is that radian measure is always used (except when otherwise indicated).

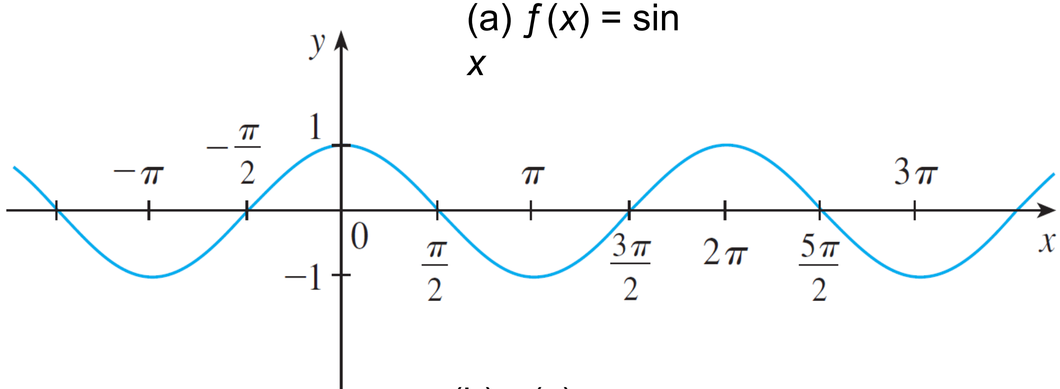
For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is $x$.

# Trigonometric Functions

Thus the graphs of the sine and cosine functions are as shown in Figure 18.




(a) $f(x) = \sin x$




(b) $g(x) = \cos x$

**Figure 18**

# Trigonometric Functions

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval [–1, 1].

Thus, for all values of *x*, we have

$$-1 \leqslant \sin x \leqslant 1 \qquad\qquad -1 \leqslant \cos x \leqslant 1$$

or, in terms of absolute values,

$$|\sin x| \leq 1 \qquad\qquad |\cos x| \leq 1$$

# Trigonometric Functions

Also, the zeros of the sine function occur at the integer multiples of $\pi$; that is,

$$\sin x = 0 \qquad \text{when} \qquad x = n\pi \qquad n \text{ an integer}$$

An important property of the sine and cosine functions is that they are periodic functions and have period $2\pi$.

This means that, for all values of $x$,

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

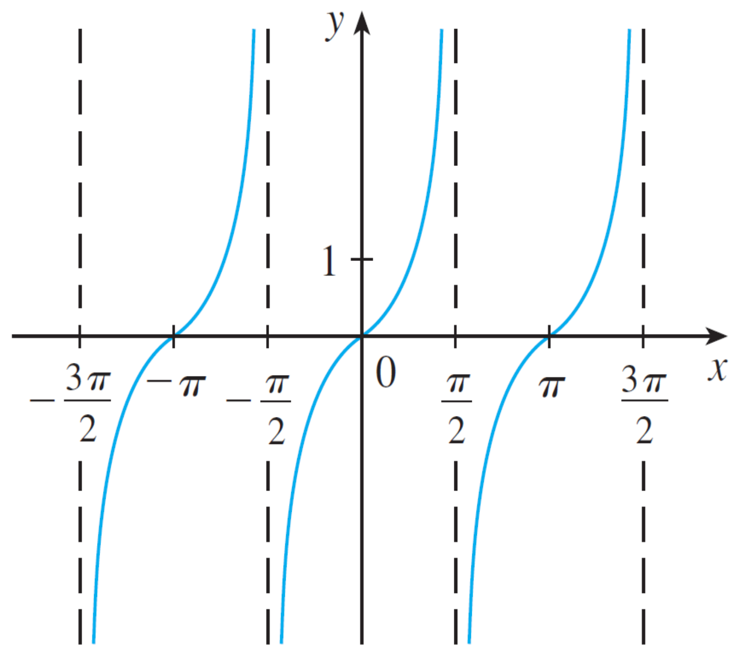
# Trigonometric Functions

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure 19. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm 3\pi/2, \ldots$.

Its range is $(-\infty, \infty)$.




$y = \tan x$

**Figure 19**

# Trigonometric Functions

Notice that the tangent function has period $\pi$:

$$\tan(x + \pi) = \tan x \qquad \text{for all } x$$

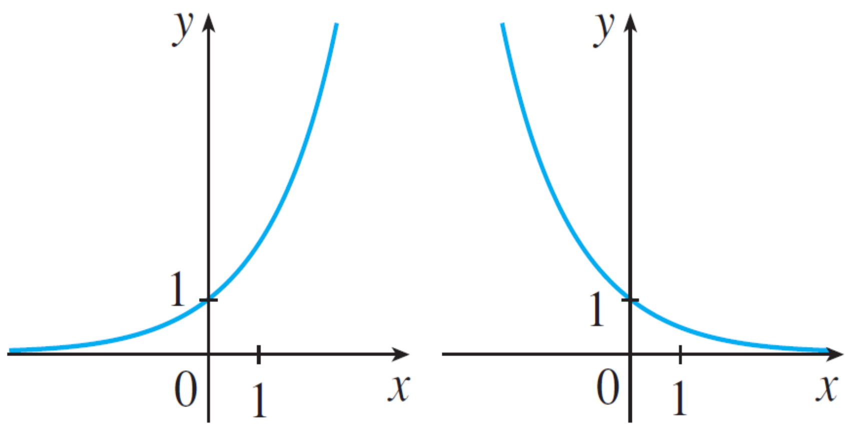
The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions.

# Exponential Functions

# Exponential Functions

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base $a$ is a positive constant.

The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 20. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.




(a) $y = 2^x$ (b) $y = (0.5)^x$

**Figure 20**

# Exponential Functions

Exponential functions are useful for modeling many natural phenomena, such as population growth (if $a > 1$) and radioactive decay (if $a < 1$).

# Logarithmic Functions

# Logarithmic Functions

The **logarithmic functions** $f(x) = \log_a x$, where the base $a$ is a positive constant, are the inverse functions of the exponential functions. Figure 21 shows the graphs of four logarithmic functions with various bases.

In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

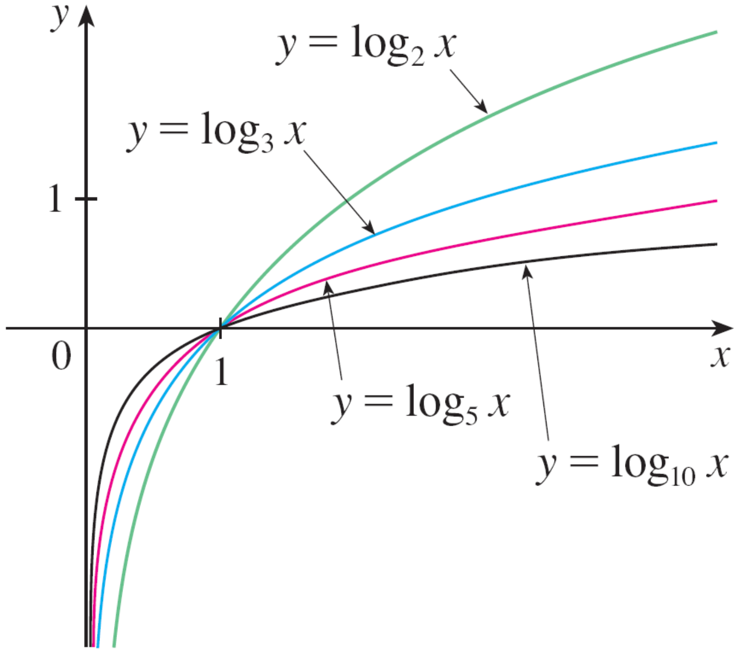



**Figure 21**

# Example 5

Classify the following functions as one of the types of functions that we have discussed.

**(a)** $f(x) = 5^x$

**(b)** $g(x) = x^5$

**(c)** $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$

**(d)** $u(t) = 1 - t + 5t^4$

# Example 5 – *Solution*

**(a)** $f(x) = 5^x$ is an exponential function.
(The $x$ is the exponent.)

**(b)** $g(x) = x^5$ is a power function. (The $x$ is the base.)
We could also consider it to be a polynomial of degree 5.

**(c)** $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ is an algebraic function.

**(d)** $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4.