

1

Functions and Models



1.3

New Functions from Old Functions



Transformations of Functions

Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions.

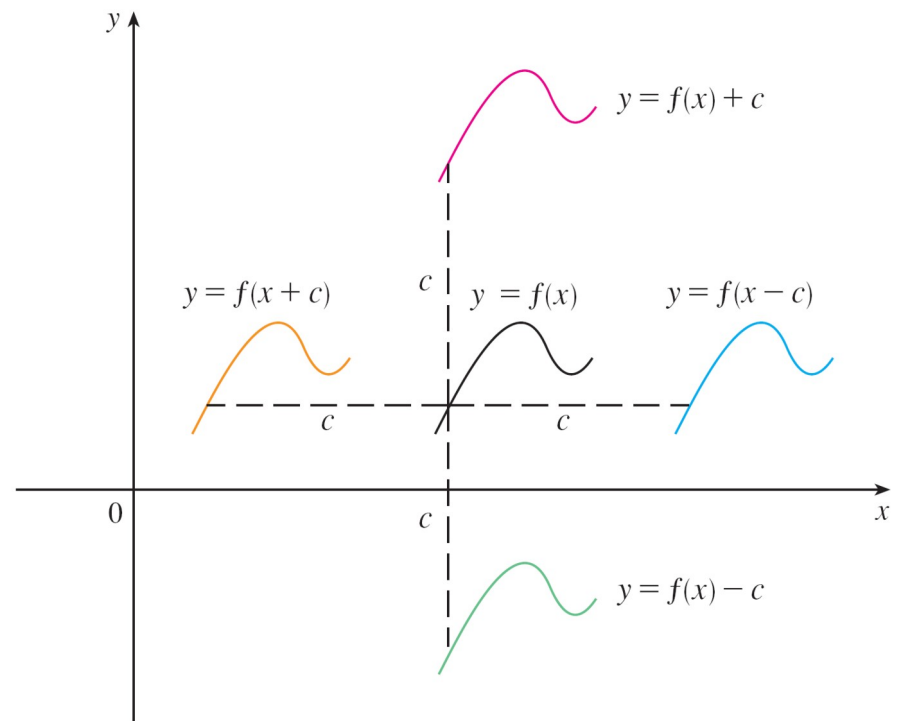
This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y -coordinate is increased by the same number c).

Transformations of Functions

Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x).

Therefore the graph of $y = f(x - c)$, is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).



Translating the graph of f
Figure 1

Transformations of Functions

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

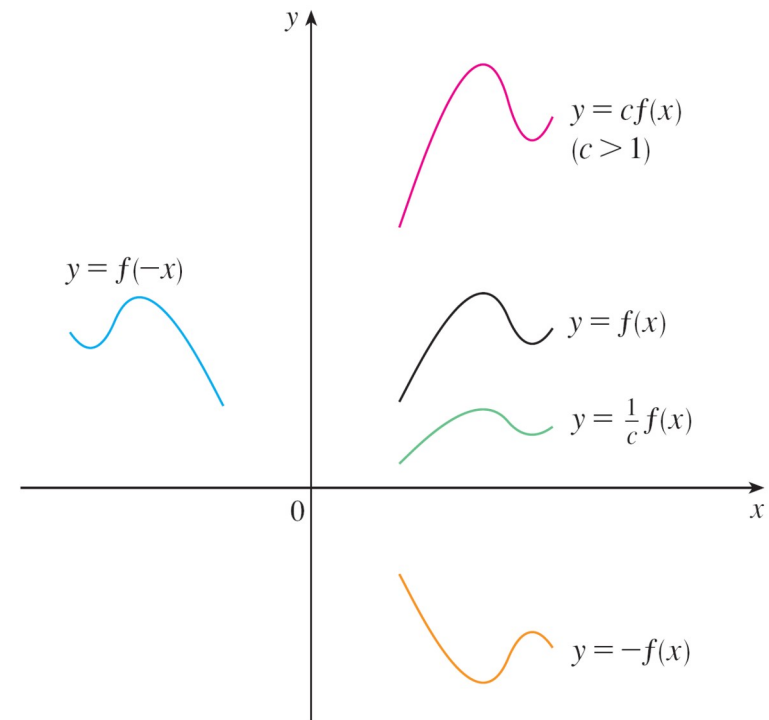
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y -coordinate is multiplied by the same number c).

Transformations of Functions

The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x -axis because the point (x, y) is replaced by the point $(x, -y)$.

(See Figure 2 and the following chart, where the results of other stretching, shrinking, and reflecting transformations are also given.)



Stretching and reflecting the graph of f

Figure 2

Transformations of Functions

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x -axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y -axis

Transformations of Functions

Figure 3 illustrates these stretching transformations when applied to the cosine function with $c = 2$.

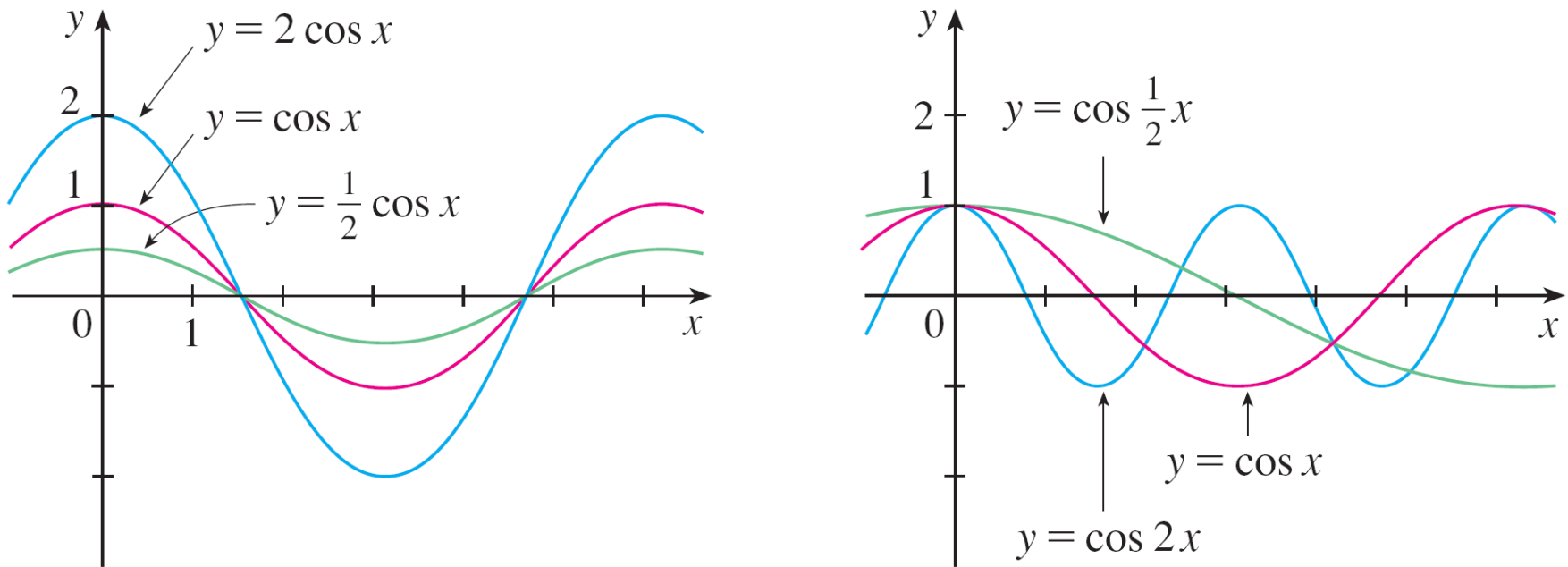


Figure 3

Transformations of Functions

For instance, in order to get the graph of $y = 2 \cos x$ we multiply the y -coordinate of each point on the graph of $y = \cos x$ by 2.

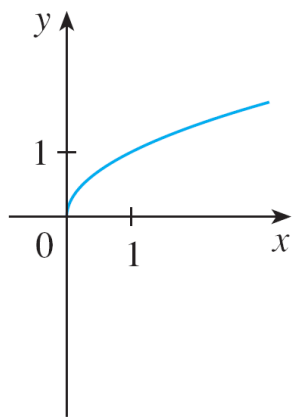
This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

Example 1

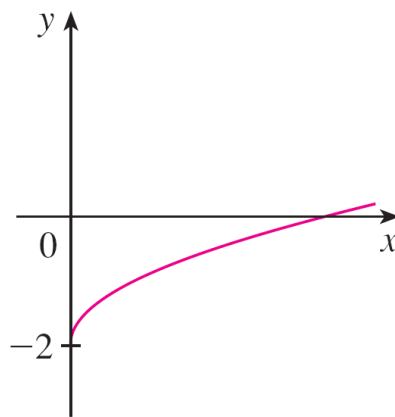
Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

Solution:

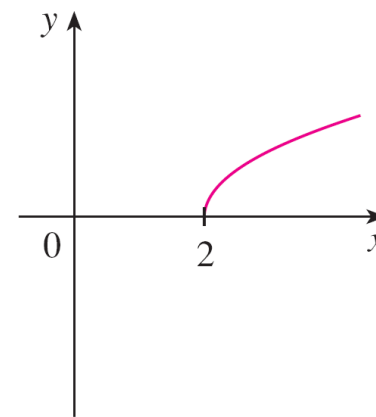
The graph of the square root function $y = \sqrt{x}$, is shown in Figure 4(a).



(a) $y = \sqrt{x}$



(b) $y = \sqrt{x} - 2$



(c) $y = \sqrt{x - 2}$

Figure 4

Example 1 – Solution

cont'd

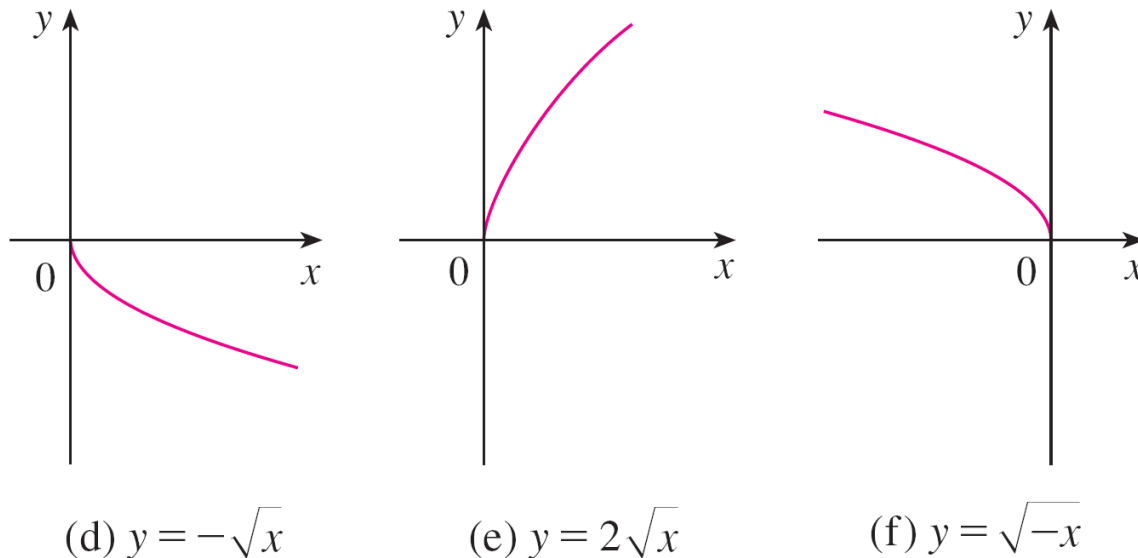


Figure 4

In the other parts of the figure we sketch $y = \sqrt{x} - 2$ by shifting 2 units downward, $y = \sqrt{x - 2}$ by shifting 2 units to the right, $y = -\sqrt{x}$ by reflecting about the x-axis, $y = 2\sqrt{x}$ by stretching vertically by a factor of 2, and $y = \sqrt{-x}$ by reflecting about the y-axis.

Transformations of Functions

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geq 0$ and $y = -f(x)$ when $f(x) < 0$.

This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: The part of the graph that lies above the x -axis remains the same; the part that lies below the x -axis is reflected about the x -axis.



Combinations of Functions

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg , and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \qquad (f - g)(x) = f(x) - g(x)$$

If the domain of f is A and the domain of g is B , then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined.

For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

Combinations of Functions

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$, but we can't divide by 0 and so the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$.

For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

Combinations of Functions

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$.

Since y is a function of u and u is, in turn, a function of x , it follows that y is ultimately a function of x . We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g .

Combinations of Functions

In general, given any two functions f and g , we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f , then we can calculate the value of $f(g(x))$.

The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f . It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (“ f circle g ”).

Definition Given two functions f and g , the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

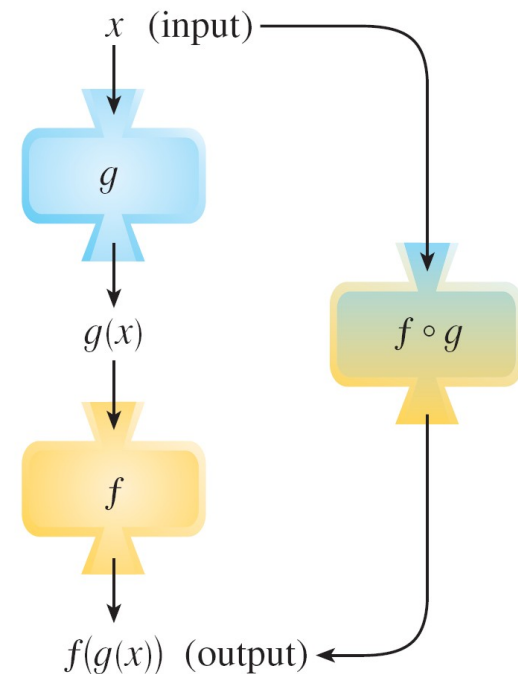
$$(f \circ g)(x) = f(g(x))$$

Combinations of Functions

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f .

In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

Figure 11 shows how to picture $f \circ g$ in terms of machines.



The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

Figure 11

Example 6

If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

Solution:

We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$= g(x^2) = x^2 - 3$$

$$(g \circ f)(x) = g(f(x))$$

Combinations of Functions

Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 6, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h , then g , and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$