

# 1

# Functions and Models



**1.5**

# **Exponential Functions**

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# Exponential Functions

The function  $f(x) = 2^x$  is called an *exponential function* because the variable,  $x$ , is the exponent. It should not be confused with the power function  $g(x) = x^2$ , in which the variable is the base.

In general, an **exponential function** is a function of the form

$$f(x) = a^x$$

where  $a$  is a positive constant. Let's recall what this means. If  $x = n$ , a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

# Exponential Functions

If  $x = 0$ , then  $a^0 = 1$ , and if  $x = -n$ , where  $n$  is a positive integer, then

$$a^{-n} = \frac{1}{a^n}$$

If  $x$  is a rational number,  $x = p/q$ , where  $p$  and  $q$  are integers and  $q > 0$ , then

$$a^x = a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

But what is the meaning of  $a^x$  if  $x$  is an irrational number? For instance, what is meant by  $2^{\sqrt{3}}$  or  $5^\pi$ ?

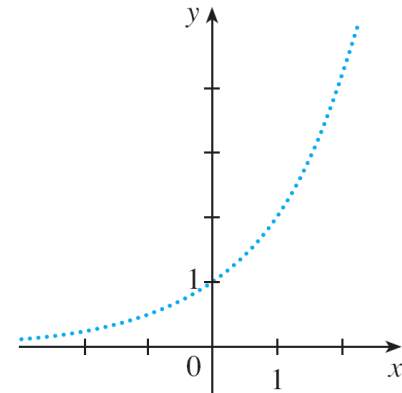
# Exponential Functions

To help us answer this question we first look at the graph of the function  $y = 2^x$ , where  $x$  is rational. A representation of this graph is shown in Figure 1.

We want to enlarge the domain of  $y = 2^x$  to include both rational and irrational numbers.

There are holes in the graph in Figure 1 corresponding to irrational values of  $x$ .

We want to fill in the holes by defining  $f(x) = 2^x$ , where  $x \in \mathbb{R}$ , so that  $f$  is an increasing function.



Representation of  $y = 2^x$ ,  $x$  rational

Figure 1

# Exponential Functions

In particular, since the irrational number  $\sqrt{3}$  satisfies

$$1.7 < \sqrt{3} < 1.8$$

we must have

$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

and we know what  $2^{1.7}$  and  $2^{1.8}$  mean because 1.7 and 1.8 are rational numbers.

# Exponential Functions

Similarly, if we use better approximations for  $\sqrt{3}$ , we obtain better approximations for  $2^{\sqrt{3}}$ .

$$1.73 < \sqrt{3} < 1.74 \quad \Rightarrow \quad 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74}$$

$$1.732 < \sqrt{3} < 1.733 \quad \Rightarrow \quad 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733}$$

$$1.7320 < \sqrt{3} < 1.7321 \quad \Rightarrow \quad 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321}$$

$$1.73205 < \sqrt{3} < 1.73206 \quad \Rightarrow \quad 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206}$$

⋮                      ⋮                      ⋮                      ⋮

# Exponential Functions

It can be shown that there is exactly one number that is greater than all of the numbers

$$2^{1.7}, 2^{1.73}, 2^{1.732}, 2^{1.7320}, 2^{1.73205}, \dots$$

and less than all of the numbers

$$2^{1.8}, 2^{1.74}, 2^{1.733}, 2^{1.7321}, 2^{1.73206}, \dots$$

We define  $2^{\sqrt{3}}$  to be this number. Using the preceding approximation process we can compute it correct to six decimal places:

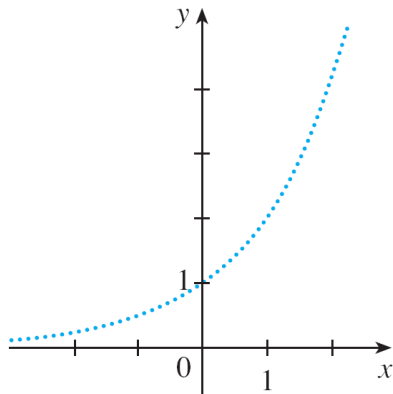
$$2^{\sqrt{3}} \approx 3.321997$$

# Exponential Functions

Similarly, we can define  $2^x$  (or  $a^x$ , if  $a > 0$ ) where  $x$  is any irrational number.

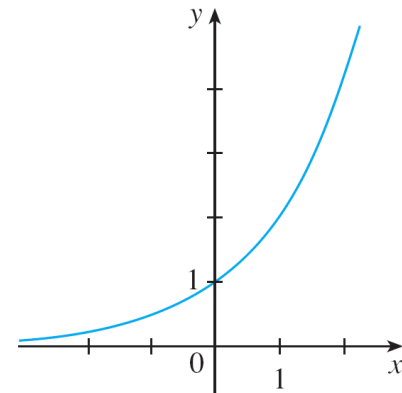
Figure 2 shows how all the holes in Figure 1 have been filled to complete the graph of the function

$$f(x) = 2^x, x \in \mathbb{R}.$$



Representation of  $y = 2^x$ ,  $x$  rational

Figure 1



$y = 2^x$ ,  $x$  real

Figure 2

# Exponential Functions

The graphs of members of the family of functions  $y = a^x$  are shown in Figure 3 for various values of the base  $a$ .

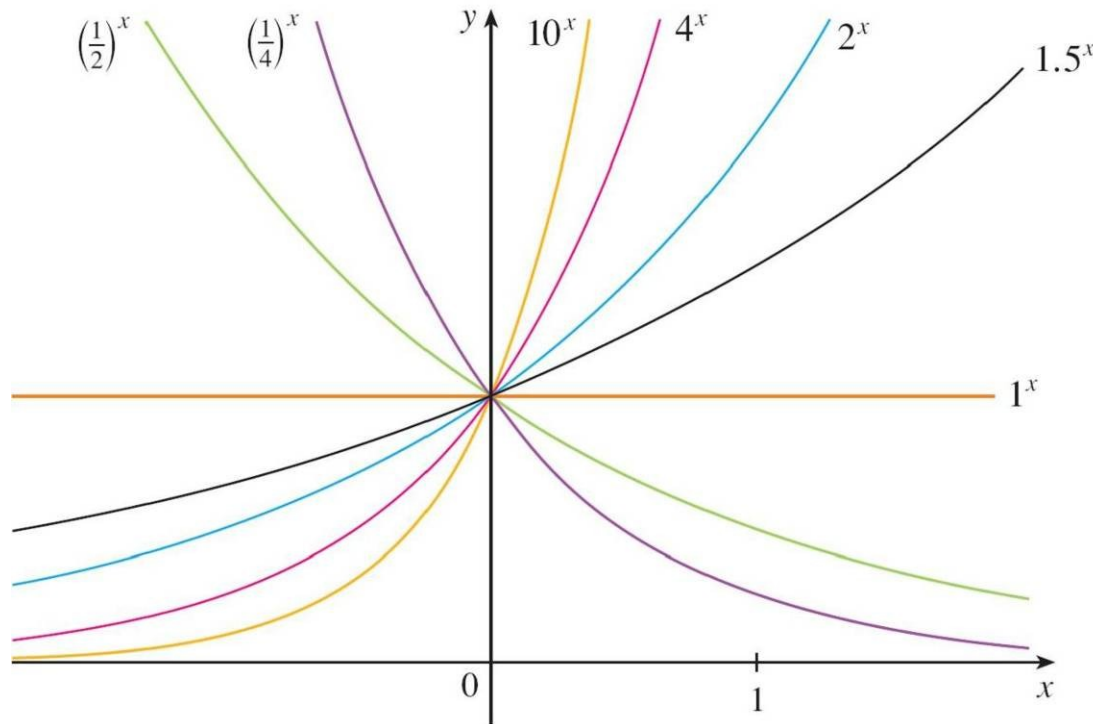


Figure 3

# Exponential Functions

Notice that all of these graphs pass through the same point  $(0, 1)$  because  $a^0 = 1$  for  $a \neq 0$ .

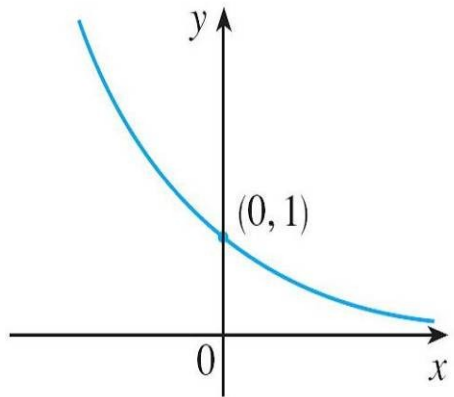
Notice also that as the base  $a$  gets larger, the exponential function grows more rapidly (for  $x > 0$ ).

You can see from Figure 3 that there are basically three kinds of exponential functions  $y = a^x$ .

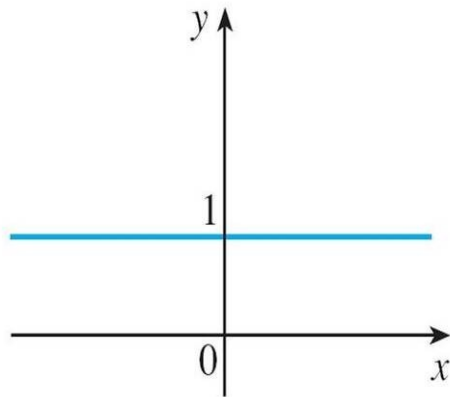
If  $0 < a < 1$ , the exponential function decreases; if  $a = 1$ , it is a constant; and if  $a > 1$ , it increases.

# Exponential Functions

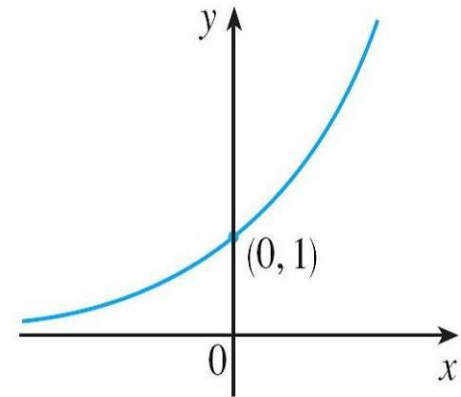
These three cases are illustrated in Figure 4.



**(a)**  $y = a^x$ ,  $0 < a < 1$



**(b)**  $y = 1^x$



**(c)**  $y = a^x$ ,  $a > 1$

**Figure 4**

# Exponential Functions

Observe that if  $a \neq 1$ , then the exponential function  $y = a^x$  has domain  $\mathbb{R}$  and range  $(0, \infty)$ .

Notice also that, since  $(1/a)^x = 1/a^x = a^{-x}$ , the graph of  $y = (1/a)^x$  is just the reflection of the graph of  $y = a^x$  about the  $y$ -axis.

# Exponential Functions

One reason for the importance of the exponential function lies in the following properties.

If  $x$  and  $y$  are rational numbers, then these laws are well known from elementary algebra. It can be proved that they remain true for arbitrary real numbers  $x$  and  $y$ .

**Laws of Exponents** If  $a$  and  $b$  are positive numbers and  $x$  and  $y$  are any real numbers, then

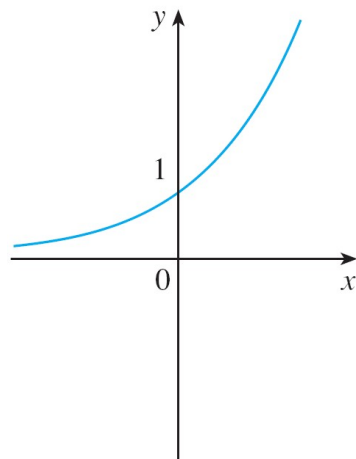
$$1. a^{x+y} = a^x a^y \quad 2. a^{x-y} = \frac{a^x}{a^y} \quad 3. (a^x)^y = a^{xy} \quad 4. (ab)^x = a^x b^x$$

# Example 1

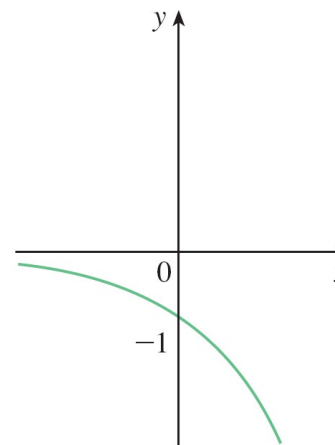
Sketch the graph of the function  $y = 3 - 2^x$  and determine its domain and range.

**Solution:**

First we reflect the graph of  $y = 2^x$  [shown in Figure 2 and Figure 5(a)] about the  $x$ -axis to get the graph of  $y = -2^x$  in Figure 5(b).



(b)  $y = 2^x$

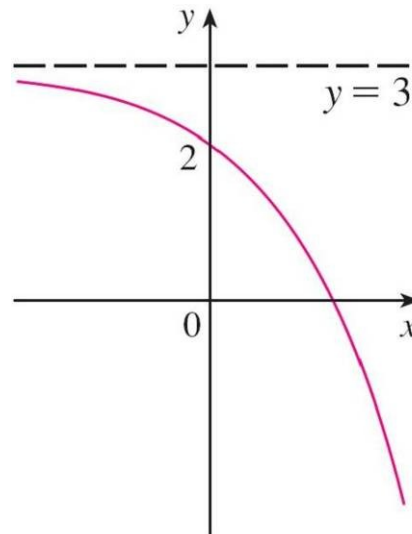


(b)  $y = -2^x$

# Example 1 – Solution

cont'd


Then we shift the graph of  $y = -2^x$  upward 3 units to obtain the graph of  $y = 3 - 2^x$  in Figure 5(c).



(c)  $y = 3 - 2^x$

Figure 5

The domain is  $\mathbb{R}$  and the range is  $(-\infty, 3)$ .



# Applications of Exponential Functions

# Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth.

First we consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour.

# Applications of Exponential Functions

If the number of bacteria at time  $t$  is  $p(t)$ , where  $t$  is measured in hours, and the initial population is  $p(0) = 1000$ , then we have

$$p(1) = 2p(0) = 2 \times 1000$$

$$p(2) = 2p(1) = 2^2 \times 1000$$

$$p(3) = 2p(2) = 2^3 \times 1000$$

It seems from this pattern that, in general,

$$p(t) = 2^t \times 1000 = (1000)2^t$$

# Applications of Exponential Functions

This population function is a constant multiple of the exponential function  $y = 2^t$ , so it exhibits the rapid growth.

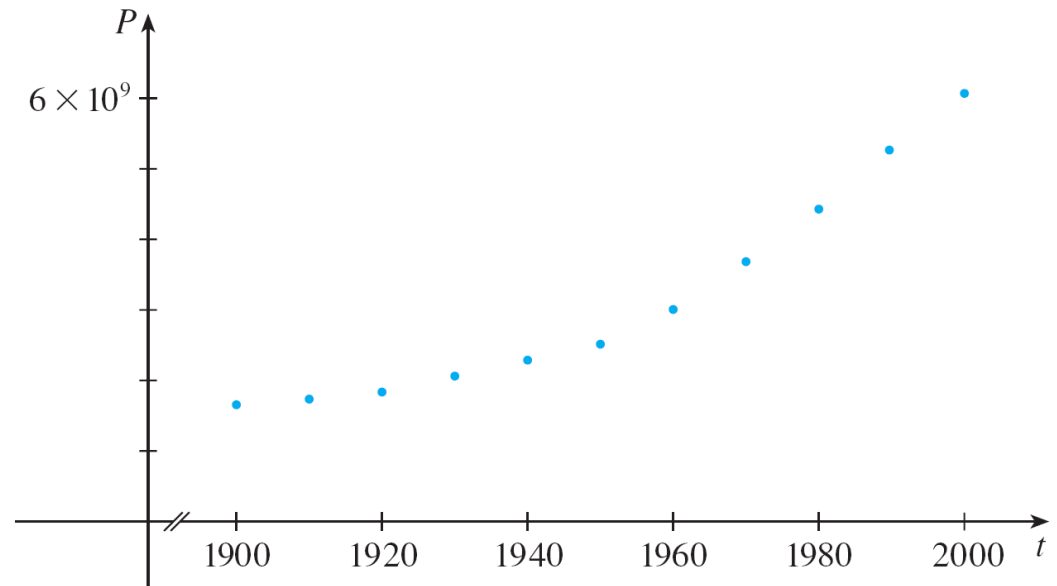
Under ideal conditions (unlimited space and nutrition and absence of disease) this exponential growth is typical of what actually occurs in nature.

# Applications of Exponential Functions

What about the human population? Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

TABLE 1

$t$	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870



Scatter plot for world population growth

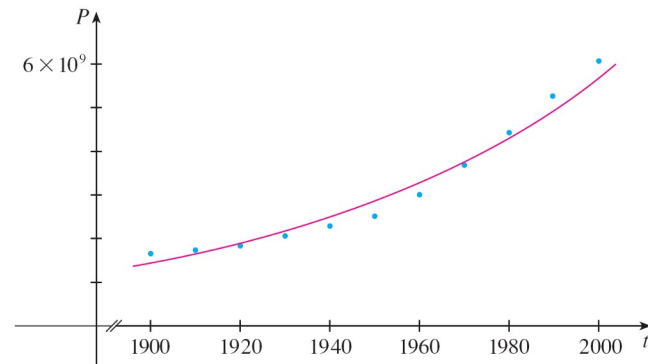
Figure 8

# Applications of Exponential Functions

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graphing calculator with exponential regression capability to apply the method of least squares and obtain the exponential model

$$P = (1436.53) \cdot (1.01395)^t$$

where  $t = 0$  corresponds to 1900. Figure 9 shows the graph of this exponential function together with the original data points.



Exponential model for population growth

**Figure 9**

# Applications of Exponential Functions

We see that the exponential curve fits the data reasonably well.

The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.



# The Number $e$

# The Number $e$

Of all possible bases for an exponential function, there is one that is most convenient for the purposes of calculus. The choice of a base  $a$  is influenced by the way the graph of  $y = a^x$  crosses the  $y$ -axis. Figures 10 and 11 show the tangent lines to the graphs of  $y = 2^x$  and  $y = 3^x$  at the point  $(0, 1)$ .

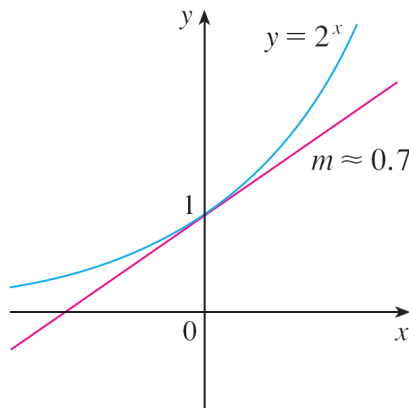


Figure 10

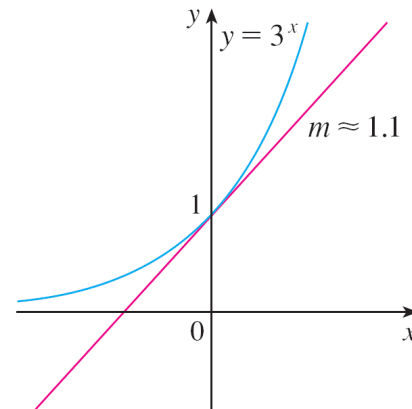


Figure 11

# The Number $e$

(For present purposes, you can think of the tangent line to an exponential graph at a point as the line that touches the graph only at that point.)

If we measure the slopes of these tangent lines at  $(0, 1)$ , we find that  $m \approx 0.7$  for  $y = 2^x$  and  $m \approx 1.1$  for  $y = 3^x$ .

# The Number $e$

It turns out that some of the formulas of calculus will be greatly simplified if we choose the base  $a$  so that the slope of the tangent line to  $y = a^x$  at  $(0, 1)$  is *exactly* 1. (See Figure 12.)

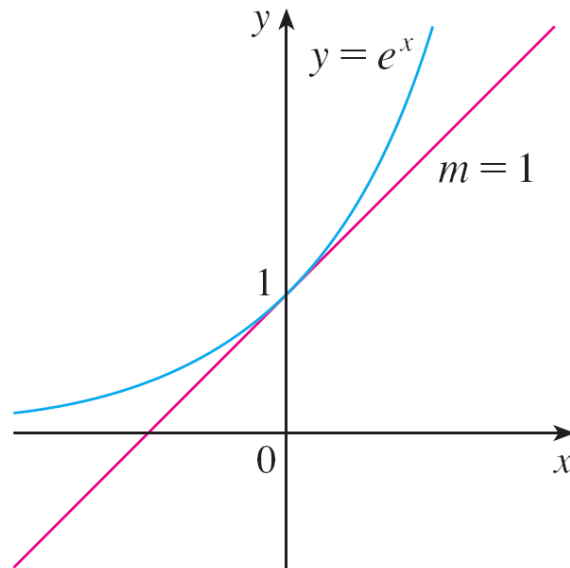


Figure 12

The natural exponential function crosses the  $y$ -axis with a slope of 1.

# The Number $e$

In fact, there *is* such a number and it is denoted by the letter  $e$ .

(This notation was chosen by the Swiss mathematician Leonhard Euler in 1727, probably because it is the first letter of the word *exponential*.)

# The Number $e$

In view of Figures 10 and 11, it comes as no surprise that the number  $e$  lies between 2 and 3 and the graph of  $y = e^x$  lies between the graphs of  $y = 2^x$  and  $y = 3^x$ . (See Figure 13.)

We will see that the value of  $e$ , correct to five decimal places, is

$$e \approx 2.71828$$

We call the function  $f(x) = e^x$  the **natural exponential function**.

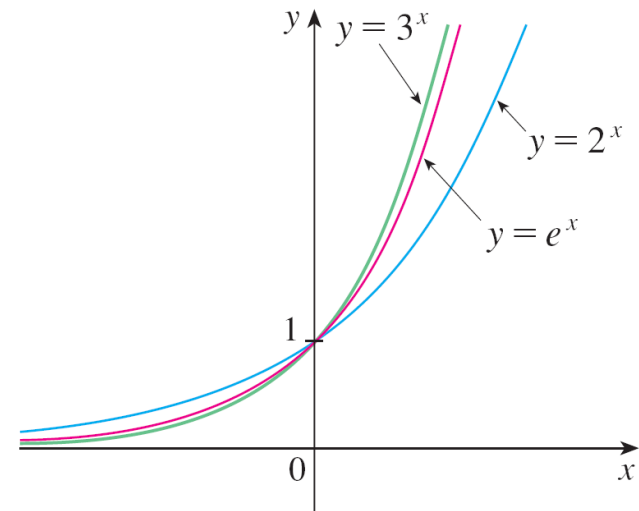


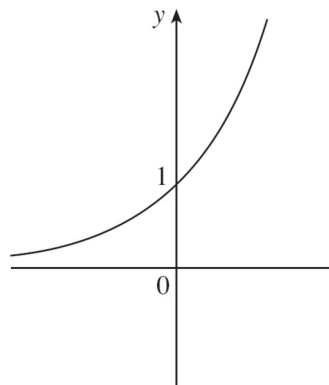
Figure 13

# Example 4

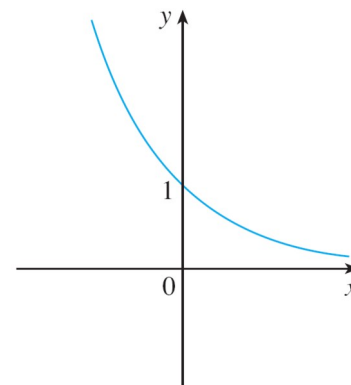
Graph the function  $y = \frac{1}{2}e^{-x} - 1$  and state the domain and range.

## Solution:

We start with the graph of  $y = e^x$  from Figures 12 and 14(a) and reflect about the  $y$ -axis to get the graph of  $y = e^{-x}$  in Figure 14(b). (Notice that the graph crosses the  $y$ -axis with a slope of  $-1$ ).



(a)  $y = e^x$



(b)  $y = e^{-x}$

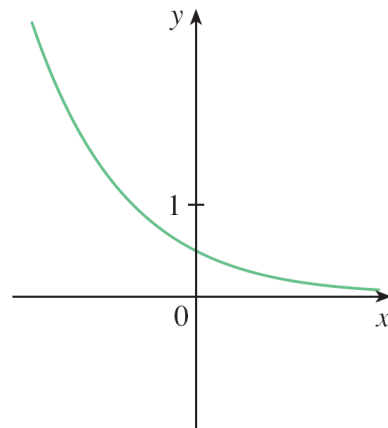
Figure 15

# Example 4 – Solution

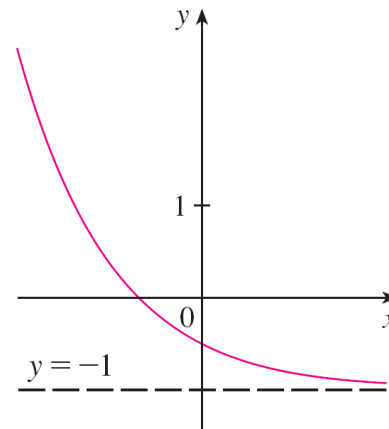
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Then we compress the graph vertically by a factor of 2 to obtain the graph of  $y = \frac{1}{2}e^{-x}$  in Figure 14(c).

Finally, we shift the graph downward one unit to get the desired graph in Figure 14(d).



(c)  $y = \frac{1}{2}e^{-x}$



(d)  $y = \frac{1}{2}e^{-x} - 1$

Figure 15

The domain is  $\mathbb{R}$  and the range is  $(-1, \infty)$ .