1

Functions and Models

1.6

Inverse Functions and Logarithms

Inverse Functions and Logarithms

Table 1 gives data from an experiment in which a bacteria culture started with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals.

The number of bacteria *N* is a function of the time *t*: $N = f(t)$.

Suppose, however, that the biologist changes her point of view and becomes interested in the time required for the population to reach various levels. In other words, she is thinking of *t* as a function of *N*.

t (hours)	$N = f(t)$ = population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

N as a function of *t* **Table 1**

Inverse Functions and Logarithms

This function is called the *inverse function* of f, denoted by f^{-1}, and read "f inverse." Thus $t = f^{-1}(N)$ is the time required for the population level to reach N.

The values of f^{-1} can be found by reading Table 1 from right to left or by consulting Table 2.

For instance, $f^{-1}(550) = 6$ because $f(6) = 550$.

Not all functions possess inverses.

N	$t = f^{-1}(N)$ = time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

t as a function of N

Table 2

Inverse Functions and Logarithms

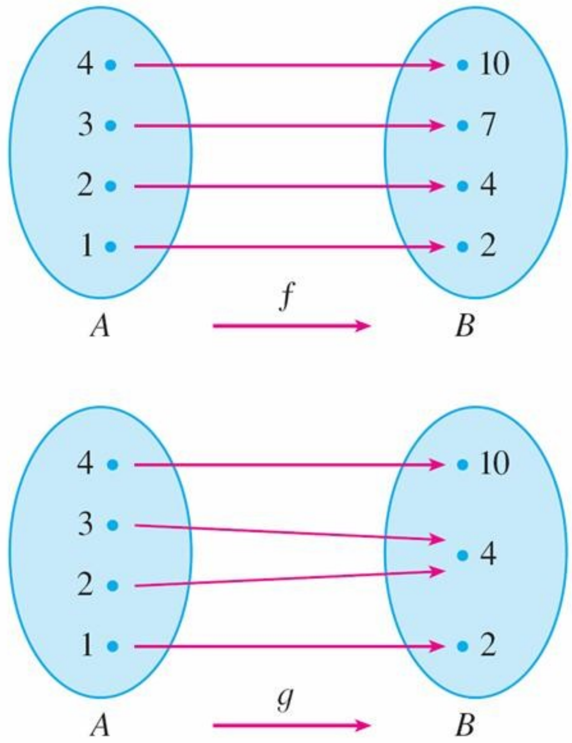
Let's compare the functions *f* and *g* whose arrow diagrams are shown in Figure 1.

Note that *f* never takes on the same value twice (any two inputs in *A* have different outputs), whereas *g* does take on the same value twice (both 2 and 3 have the same output, 4).

In symbols,

$$g(2) = g(3)$$

but $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$



f is one-to-one; *g* is not

Figure 1

Inverse Functions and Logarithms

Functions that share this property with *f* are called *one-to-one functions.*

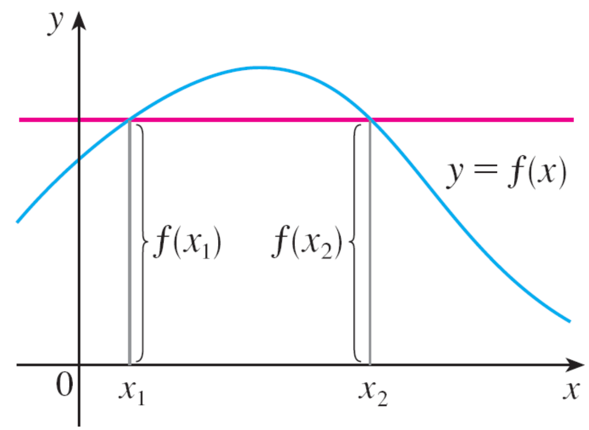
1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Inverse Functions and Logarithms

If a horizontal line intersects the graph of *f* in more than one point, then we see from Figure 2 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$.

This means that *f* is not one-to-one.



This function is not one-to-one because $f(x_1) = f(x_2)$.

Figure 2

Therefore we have the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 1

Is the function $f(x) = x^3$ one-to-one?

Solution 1:

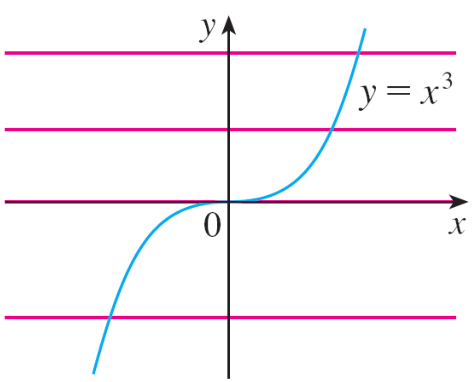
If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers can't have the same cube).

Therefore, by Definition 1, $f(x) = x^3$ is one-to-one.

Solution 2:

From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once.

Therefore, by the Horizontal Line Test, f is one-to-one.



$f(x) = x^3$ is one-to-one.

Figure 3

Inverse Functions and Logarithms

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

This definition says that if f maps x into y, then f^{-1} maps y back into x. (If f were not one-to-one, then f^{-1} would not be uniquely defined.)

Inverse Functions and Logarithms

The arrow diagram in Figure 5 indicates that f^{-1} reverses the effect of f.

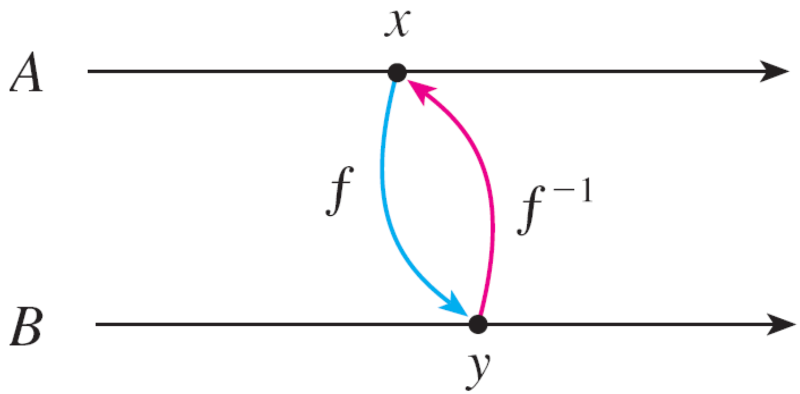


Figure 5

Note that

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

Inverse Functions and Logarithms

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

Caution

Do not mistake the –1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \text{ does } \textit{not} \text{ mean } \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ could, however, be written as $[f(x)]^{-1}$.

Example 3

If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

Solution:

From the definition of f^{-1} we have

$f^{-1}(7) = 3$	because	$f(3) = 7$
$f^{-1}(5) = 1$	because	$f(1) = 5$
$f^{-1}(-10) = 8$	because	$f(8) = -10$

Example 3 – *Solution*

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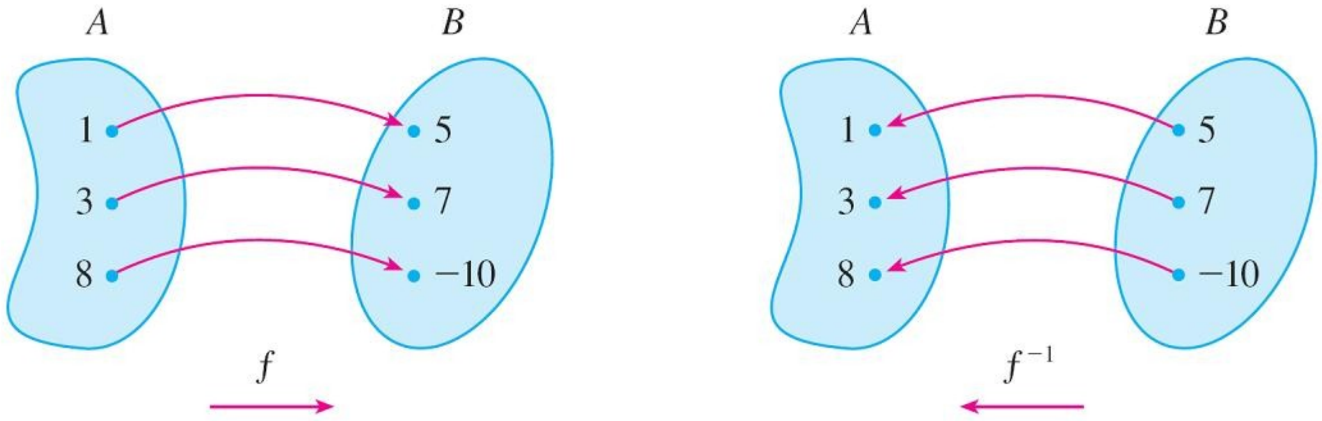
The diagram in Figure 6 makes it clear how f^{-1} reverses the effect of f in this case.



The inverse function reverses inputs and outputs.

Figure 6

Inverse Functions and Logarithms

The letter *x* is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on *f*, we usually reverse the roles of *x* and *y* in Definition 2 and write

3

$$f^{-1}(x) = y \iff f(y) = x$$

By substituting for *y* in Definition 2 and substituting for *x* in 3 we get the following **cancellation equations**:

4

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Inverse Functions and Logarithms

The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (see the machine diagram in Figure 7).

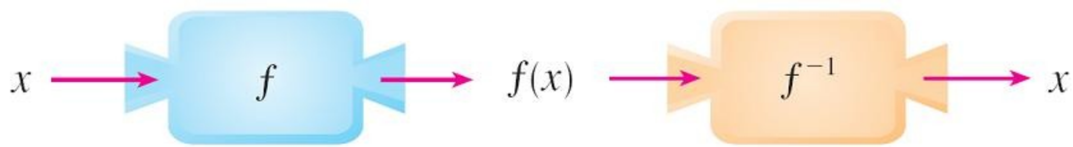


Figure 7

Thus f^{-1} undoes what f does.

The second equation says that f undoes what f^{-1} does.

Inverse Functions and Logarithms

For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and so the cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$

$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply say that the cube function and the cube root function cancel each other when applied in succession.

Inverse Functions and Logarithms

Now let's see how to compute inverse functions.

If we have a function $y = f(x)$ and are able to solve this equation for x in terms of y, then according to Definition 2 we must have $x = f^{-1}(y)$.

If we want to call the independent variable x, we then interchange x and y and arrive at the equation $y = f^{-1}(x)$.

5 How to Find the Inverse Function of a One-to-One Function f

Step 1 Write $y = f(x)$.

Step 2 Solve this equation for x in terms of y (if possible).

Step 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Inverse Functions and Logarithms

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f.

Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}.

But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

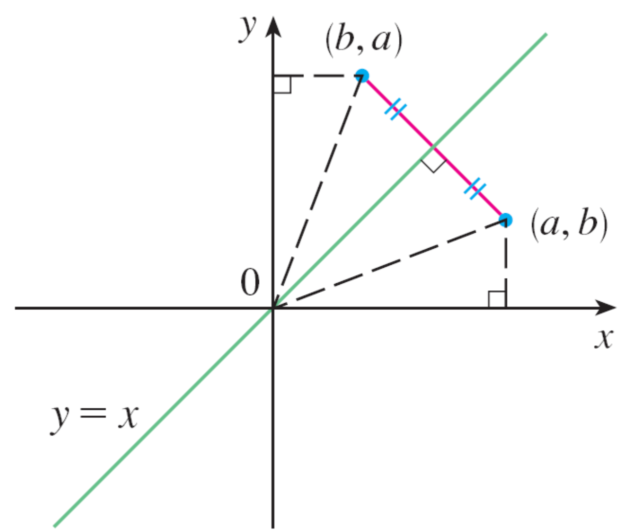


Figure 8

Inverse Functions and Logarithms

Therefore, as illustrated by Figure 9:

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

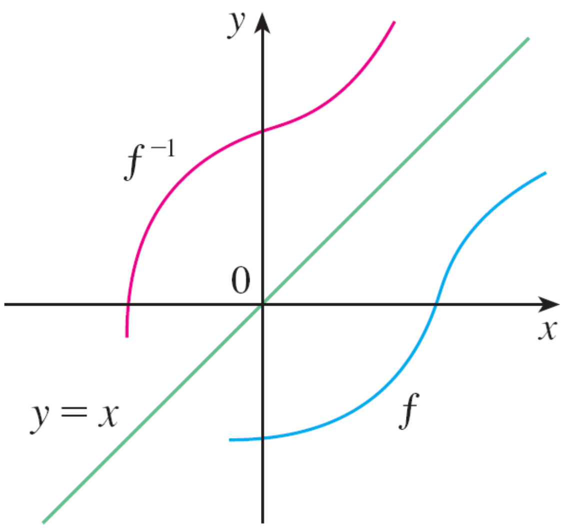


Figure 9

Logarithmic Functions

Logarithmic Functions

If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. It therefore has an inverse function f^{-1}, which is called the **logarithmic function with base *a*** and is denoted by $\log_a$.

If we use the formulation of an inverse function given by [3],

$$f^{-1}(x) = y \iff f(y) = x$$

then we have

6

$$\log_a x = y \iff a^y = x$$

Logarithmic Functions

Thus, if $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

For example, $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.

The cancellation equations $\boxed{4}$, when applied to the functions $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, become

7

$$\log_a(a^x) = x \quad \text{for every } x \in \mathbb{R}$$

$$a^{\log_a x} = x \quad \text{for every } x > 0$$

Logarithmic Functions

The logarithmic function $\log_a$ has domain $(0, \infty)$ and range $\mathbb{R}$. Its graph is the reflection of the graph of $y = a^x$ about the line $y = x$.

Figure 11 shows the case where $a > 1$. (The most important logarithmic functions have base $a > 1$.)

The fact that $y = a^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_a x$ is a very slowly increasing function for $x > 1$.

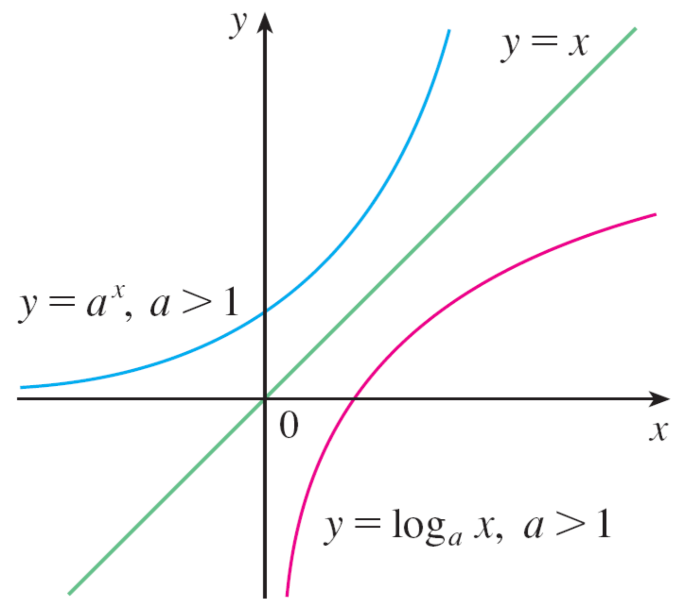


Figure 11

Logarithmic Functions

Figure 12 shows the graphs of $y = \log_a x$ with various values of the base $a > 1$.

Since $\log_a 1 = 0$, the graphs of all logarithmic functions pass through the point (1, 0).

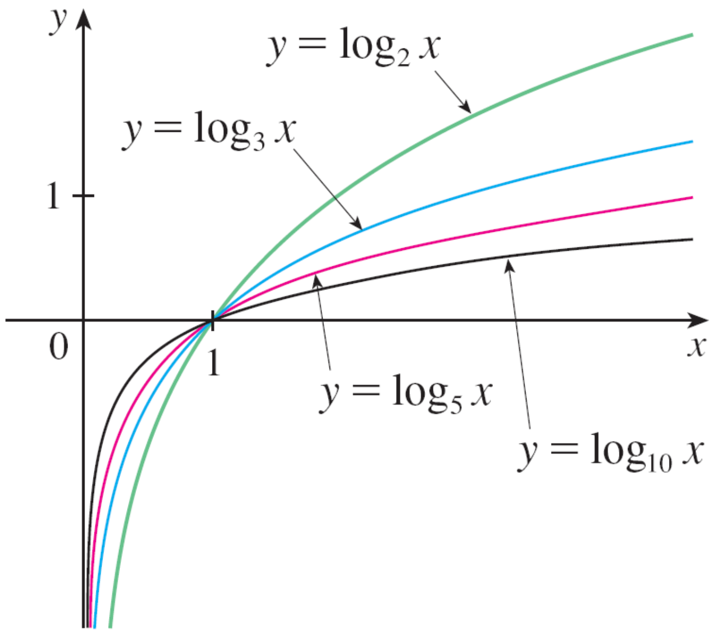


Figure 12

Logarithmic Functions

The following properties of logarithmic functions follow from the corresponding properties of exponential functions.

Laws of Logarithms If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + \log_a y$

2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a(x^r) = r \log_a x$ (where r is any real number)

Example 6

Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

Solution:

Using Law 2, we have

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$$

$$= \log_2 16$$

$$= 4$$

because $2^4 = 16$.

Natural Logarithms

Natural Logarithms

Of all possible bases *a* for logarithms, we will see that the most convenient choice of a base is the number *e*.

The logarithm with base *e* is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

If we put *a* = *e* and replace $\log_e$ with "ln" in [6] and [7], then the defining properties of the natural logarithm function become

8

$$\ln x = y \iff e^y = x$$

Natural Logarithms

9

$$\ln(e^{x}) = x \qquad x \in \mathbb{R}$$

$$e^{\ln x} = x \qquad x > 0$$

In particular, if we set $x = 1$, we get

$$\ln e = 1$$

Example 7

Find x if $\ln x = 5$.

Solution 1:

From [8] we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore, $x = e^5$.

(If you have trouble working with the "ln" notation, just replace it by $\log_e$. Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

Example 7 – *Solution*

cont'd

Solution 2:

Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in $\boxed{9}$ says that $e^{\ln x} = x$.

Therefore $x = e^5$.

Natural Logarithms

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

10 **Change of Base Formula** For any positive number a ($a \neq 1$), we have

$$\log_a x = \frac{\ln x}{\ln a}$$

Example 10

Evaluate $\log_8 5$ correct to six decimal places.

Solution:

Formula 10 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8}$$

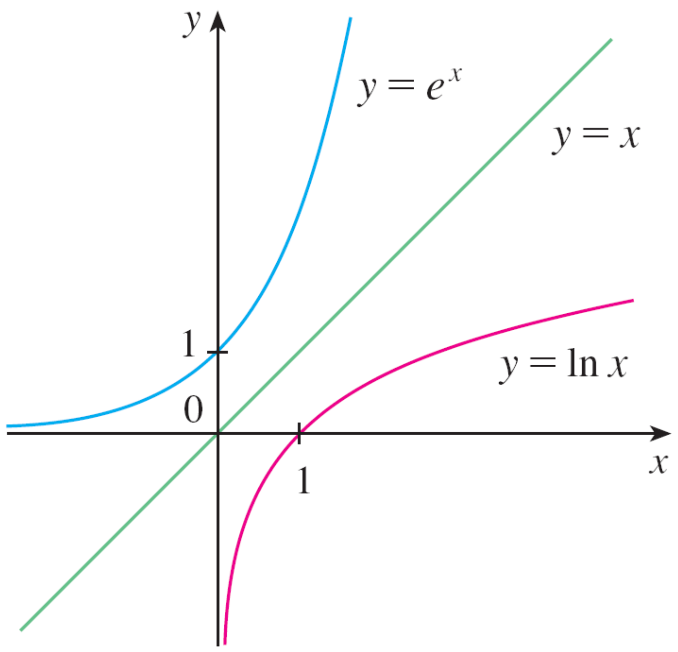
$$\approx 0.773976$$

Graph and Growth of the Natural Logarithm

Graph and Growth of the Natural Logarithm

The graphs of the exponential function $y = e^x$ and its inverse function, the natural logarithm function, are shown in Figure 13.

Because the curve $y = e^x$ crosses the y-axis with a slope of 1, it follows that the reflected curve $y = \ln x$ crosses the x-axis with a slope of 1.



The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$

Figure 13

Graph and Growth of the Natural Logarithm

In common with all other logarithmic functions with base greater than 1, the natural logarithm is an increasing function defined on $(0, \infty)$ and the *y*-axis is a vertical asymptote.

(This means that the values of $\ln x$ become very large negative as x approaches 0.)

Example 11

Sketch the graph of the function $y = \ln(x - 2) - 1$.

Solution:

We start with the graph of $y = \ln x$ as given in Figure 13.

We shift it 2 units to the right to get the graph of $y = \ln(x - 2)$ and then we shift it 1 unit downward to get the graph of $y = \ln(x - 2) - 1$. (See Figure 14.)

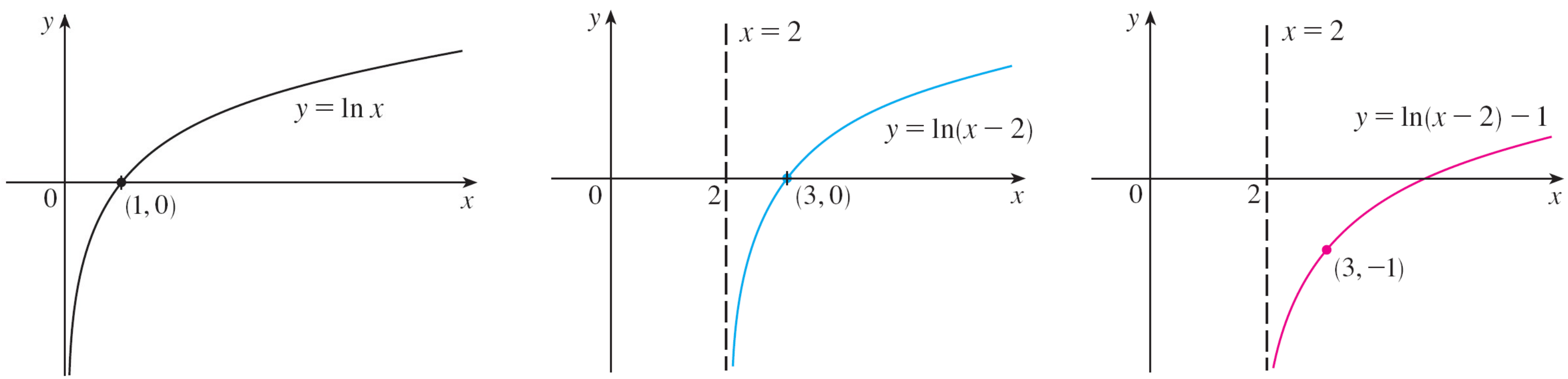


Figure 14

Graph and Growth of the Natural Logarithm

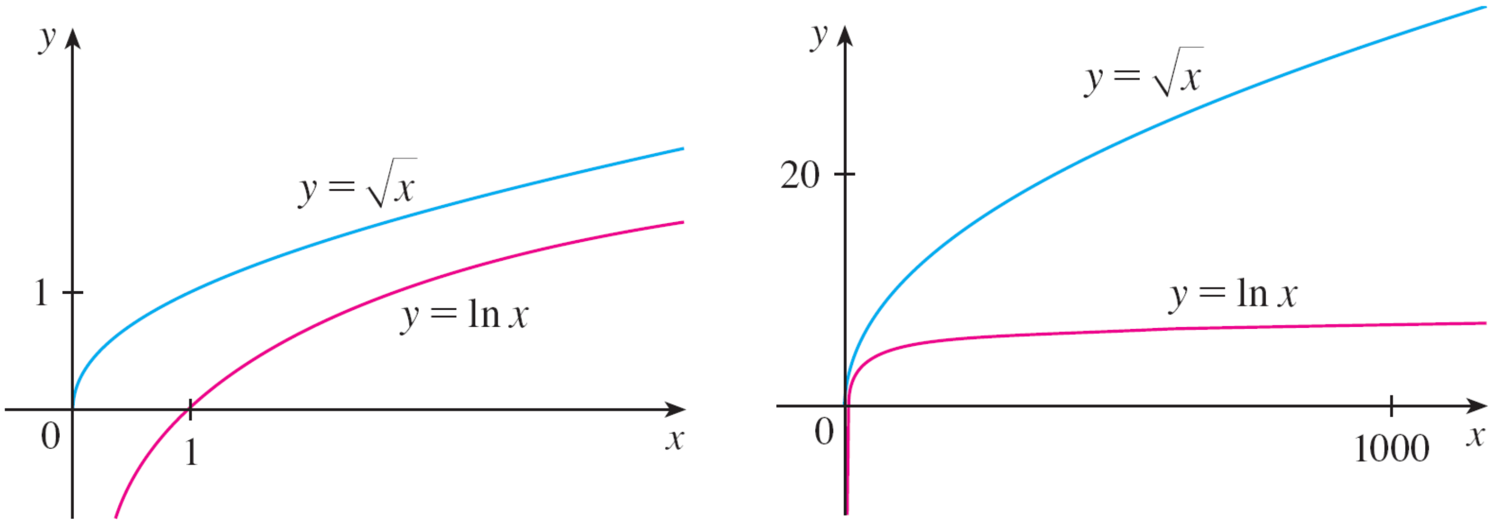
Although ln *x* is an increasing function, it grows *very* slowly when $x > 1$. In fact, ln *x* grows more slowly than any positive power of *x*.

To illustrate this fact, we compare approximate values of the functions $y = \ln x$ and $y = x^{1/2} = \sqrt{x}$ in the following table.

x	1	2	5	10	50	100	500	1000	10,000	100,000
$\ln x$	0	0.69	1.61	2.30	3.91	4.6	6.2	6.9	9.2	11.5
$\sqrt{x}$	1	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316
$\frac{\ln x}{\sqrt{x}}$	0	0.49	0.72	0.73	0.55	0.46	0.28	0.22	0.09	0.04

Graph and Growth of the Natural Logarithm

We graph them in Figures 15 and 16.



Figures 15 **Figures 16**

You can see that initially the graphs of y = $\sqrt{x}$ and *y* = ln *x* grow at comparable rates, but eventually the root function far surpasses the logarithm.