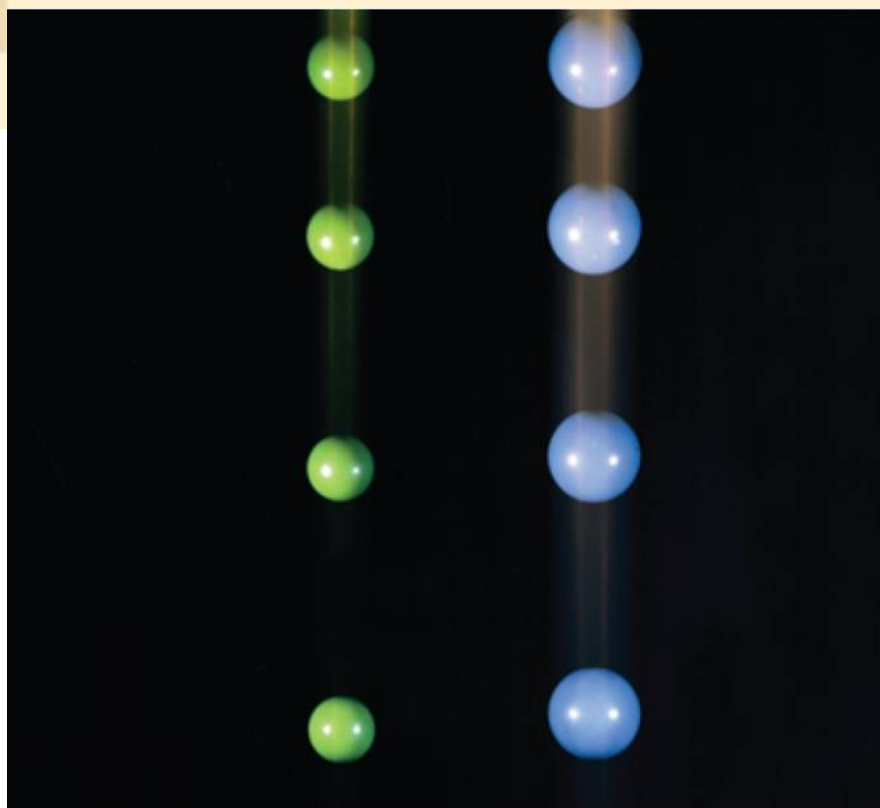


2

Limits and Derivatives



2.3

Calculating Limits Using the Limit Laws

Calculating Limits Using the Limit Laws

In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) \quad \text{and} \quad \lim_{x \rightarrow a} g(x)$$

exist. Then

$$1. \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$2. \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

$$4. \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$5. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0$$

Calculating Limits Using the Limit Laws

These five laws can be stated verbally as follows:

Sum Law
limits.

1. The limit of a sum is the sum of the

Difference Law

difference of the limits.

2. The limit of a difference is the

Constant Multiple Law

function is the constant times the
the function.

3. The limit of a constant times a
limit of

Calculating Limits Using the Limit Laws

Product Law

of the limits.

4. The limit of a product is the product

Quotient Law

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

For instance, if $f(x)$ is close to L and $g(x)$ is close to M , it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$.

Example 1

Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \rightarrow -2} [f(x) + 5g(x)]$ (b) $\lim_{x \rightarrow 1} [f(x)g(x)]$ (c) $\lim_{x \rightarrow 2} \frac{f(x)}{g(x)}$

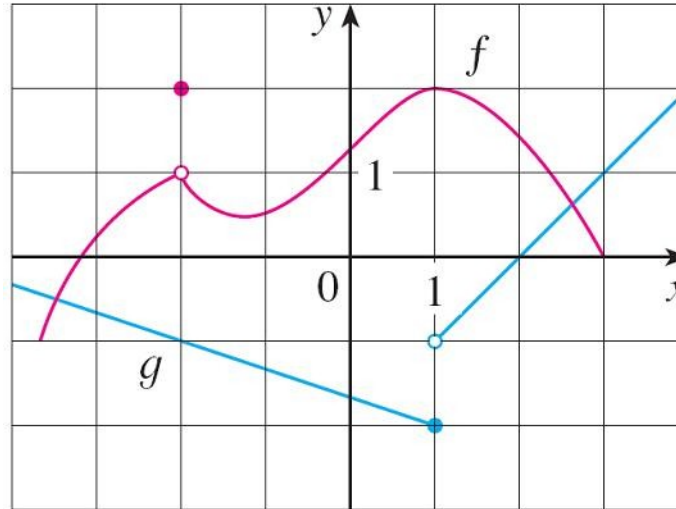


Figure 1

Example 1(a) – *Solution*

From the graphs of f and g we see that

$$\lim_{x \rightarrow -2} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow -2} g(x) = -1$$

Therefore we have

$$\lim_{x \rightarrow -2} [f(x) + 5g(x)] = \lim_{x \rightarrow -2} f(x) + \lim_{x \rightarrow -2} [5g(x)] \quad (\text{by Law 1})$$

$$= \lim_{x \rightarrow -2} f(x) + 5 \lim_{x \rightarrow -2} g(x) \quad (\text{by Law 3})$$

$$= 1 + 5(-1)$$

$$= -4$$

Example 1(b) – *Solution*

cont'd

We see that $\lim_{x \rightarrow 1} f(x) = 2$. But $\lim_{x \rightarrow 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x \rightarrow 1^-} g(x) = -2 \qquad \lim_{x \rightarrow 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we *can* use Law 4 for the one-sided limits:

$$\lim_{x \rightarrow 1^-} [f(x)g(x)] = 2 \cdot (-2) = -4 \qquad \lim_{x \rightarrow 1^+} [f(x)g(x)] = 2 \cdot (-1) = -2$$

The left and right limits aren't equal, so $\lim_{x \rightarrow 1} [f(x)g(x)]$ does not exist.

Example 1(c) – Solution

cont'd

The graphs show that

$$\lim_{x \rightarrow 2} f(x) \approx 1.4 \quad \text{and} \quad \lim_{x \rightarrow 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5.

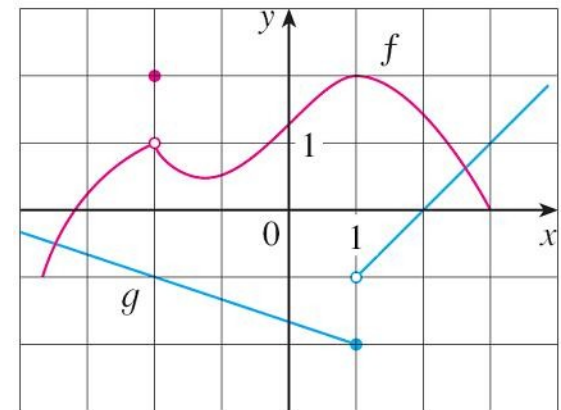


Figure 1

The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

Calculating Limits Using the Limit Laws

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

$$6. \lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n \quad \text{where } n \text{ is a positive integer}$$

In applying these six limit laws, we need to use two special limits:

$$7. \lim_{x \rightarrow a} c = c$$

$$8. \lim_{x \rightarrow a} x = a$$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$).

Calculating Limits Using the Limit Laws

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

$$9. \lim_{x \rightarrow a} x^n = a^n \quad \text{where } n \text{ is a positive integer}$$

A similar limit holds for roots as follows.

$$10. \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{where } n \text{ is a positive integer}$$

(If n is even, we assume that $a > 0$.)

More generally, we have the following law.

Root Law

$$11. \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \quad \text{where } n \text{ is a positive integer}$$

[If n is even, we assume that $\lim_{x \rightarrow a} f(x) > 0$.]

Calculating Limits Using the Limit Laws

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a* .

In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, provided the limits exist.

Calculating Limits Using the Limit Laws

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

Calculating Limits Using the Limit Laws

The next two theorems give two additional properties of limits.

2 Theorem If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

Calculating Limits Using the Limit Laws

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7.

It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a , and if f and h have the same limit L at a , then g is forced to have the same limit L at a .

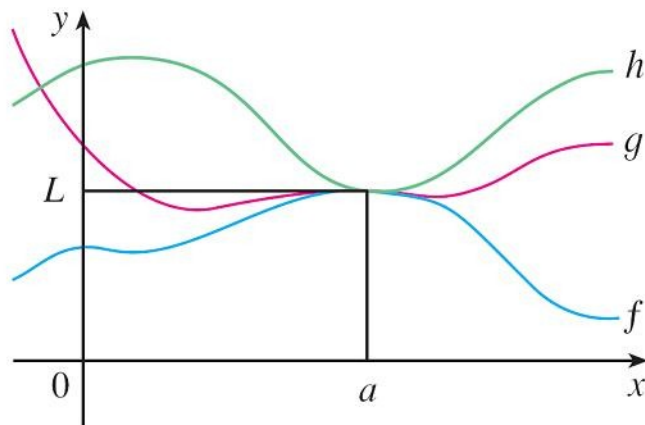


Figure 7