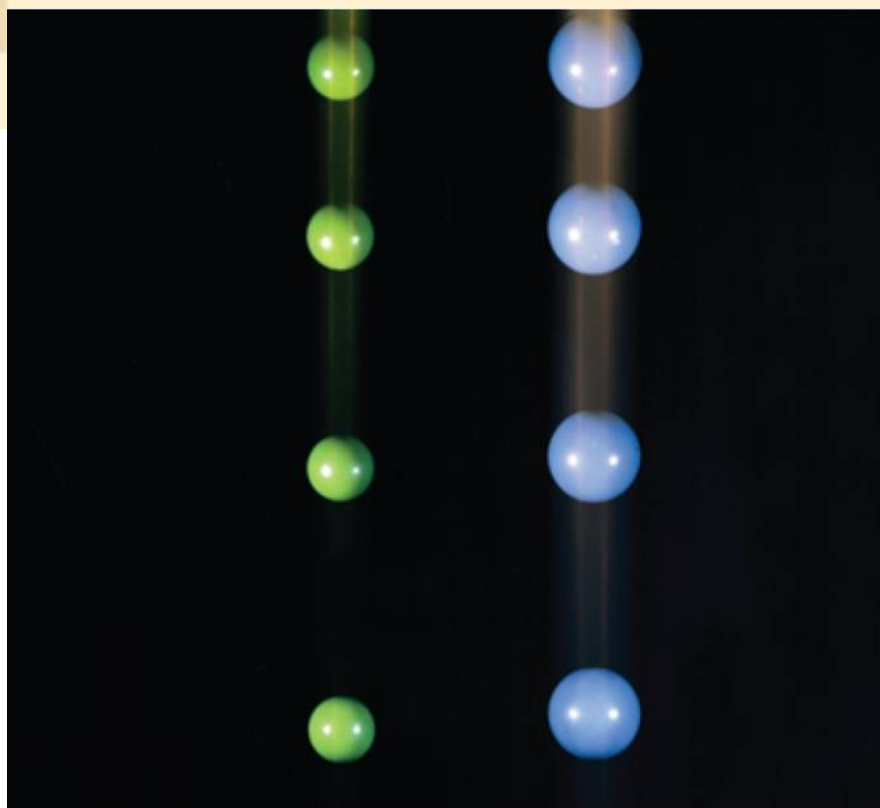


2

Limits and Derivatives



2.8

The Derivative as a Function

The Derivative as a Function

We have considered the derivative of a function f at a fixed number a :

$$\boxed{1} \quad f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x , we obtain

$$\boxed{2} \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

The Derivative as a Function

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of f** and defined by Equation 2.

We know that the value of f' at x , $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The function f' is called the derivative of f because it has been “derived” from f by the limiting operation in Equation 2. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f .

Example 1

The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f' .

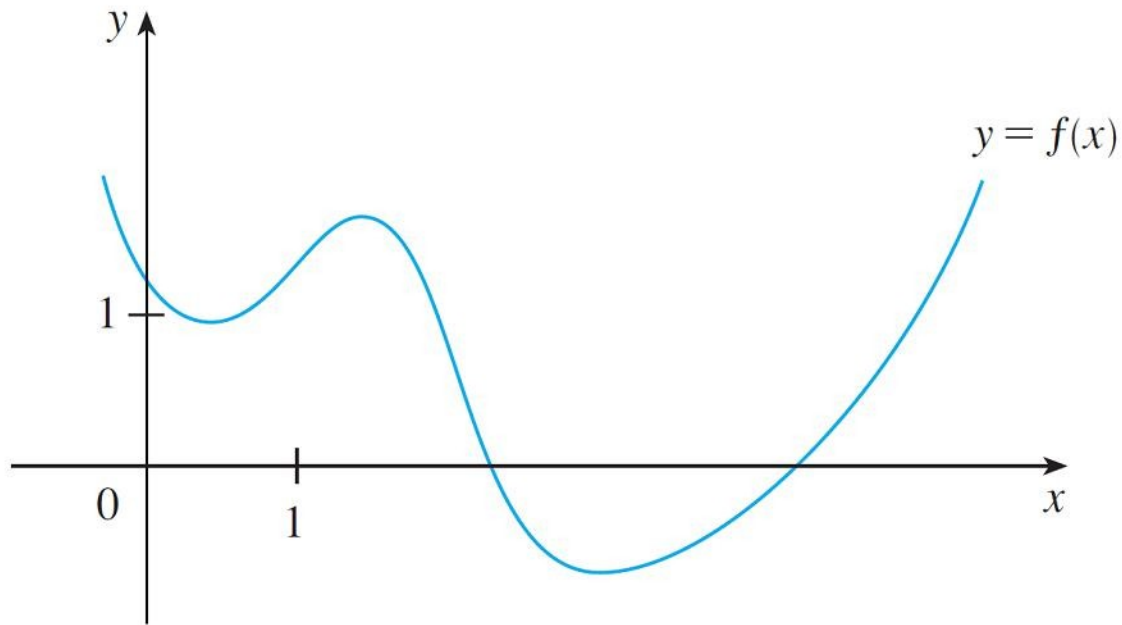


Figure 1

Example 1 – Solution

We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for $x = 5$ we draw the tangent at P in Figure 2(a) and estimate its slope to be about 1.5 , so $f'(5) \approx 1.5$.

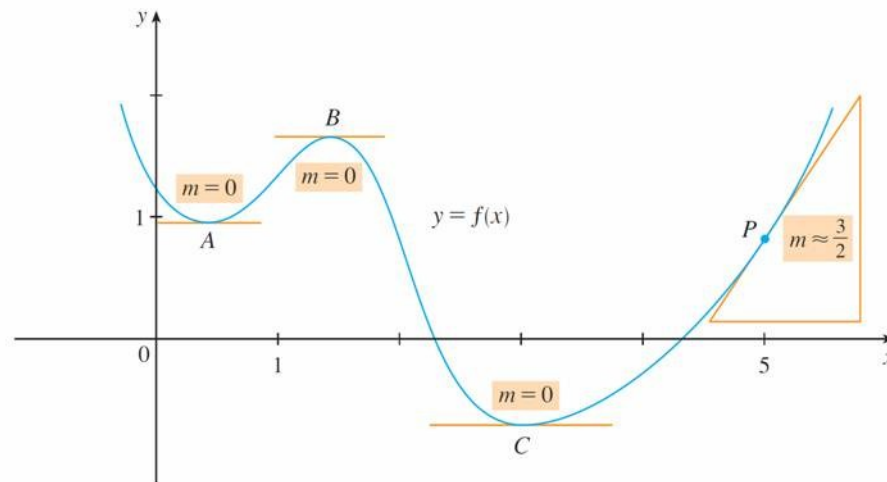


Figure 2(a)

Example 1 – Solution

cont'd

This allows us to plot the point $P'(5, 1.5)$ on the graph of f' directly beneath P . Repeating this procedure at several points, we get the graph shown in Figure 2(b).

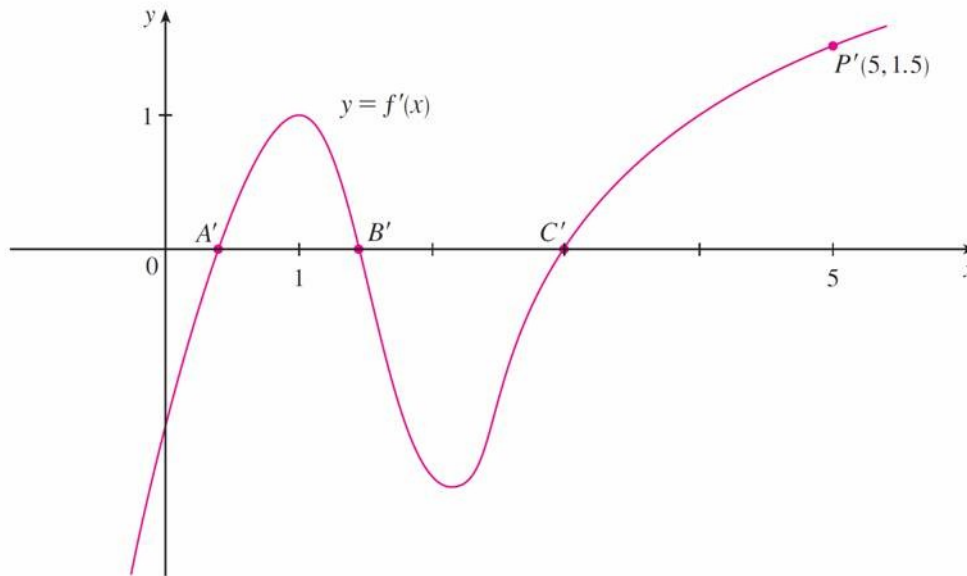


Figure 2(b)

Example 1 – *Solution*

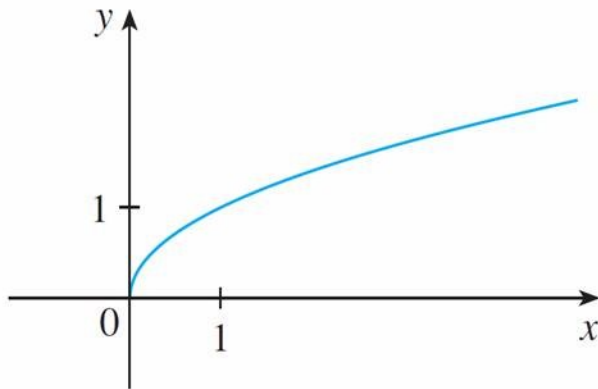
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Notice that the tangents at A , B , and C are horizontal, so the derivative is 0 there and the graph of f' crosses the x -axis at the points A' , B' , and C' , directly beneath A , B , and C .

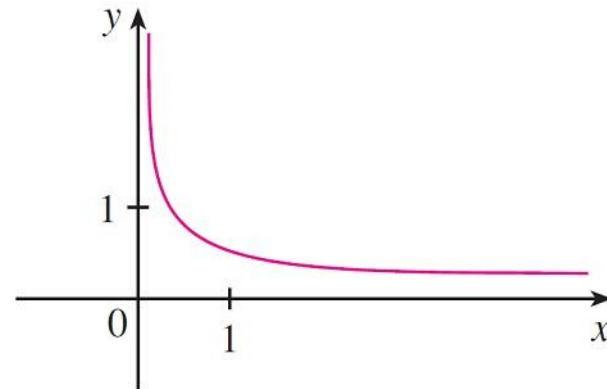
Between A and B the tangents have positive slope, so $f'(x)$ is positive there. But between B and C the tangents have negative slope, so $f'(x)$ is negative there.

The Derivative as a Function

When x is close to 0, \sqrt{x} is also close to 0, so $f'(x) = 1/(2\sqrt{x})$ is very large and this corresponds to the steep tangent lines near $(0, 0)$ in Figure 4(a) and the large values of $f'(x)$ just to the right of 0 in Figure 4(b).



(a) $f(x) = \sqrt{x}$



(b) $f'(x) = \frac{1}{2\sqrt{x}}$

Figure 4

The Derivative as a Function

When x is large, $f'(x)$ is very small and this corresponds to the flatter tangent lines at the far right of the graph of f and the horizontal asymptote of the graph of f' .



Other Notations

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y , then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

Other Notations

The symbol dy/dx , which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation.

We can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

Other Notations

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a , we use the notation

$$\left. \frac{dy}{dx} \right|_{x=a} \quad \text{or} \quad \left. \frac{dy}{dx} \right]_{x=a}$$

which is a synonym for $f'(a)$.

3 Definition A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval (a, b)** [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 5

Where is the function $f(x) = |x|$ differentiable?

Solution:

If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{|x + h| - |x|}{h} = \lim_{h \rightarrow 0} \frac{(x + h) - x}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1 \end{aligned}$$

and so f is differentiable for any $x > 0$.

Example 5 – Solution

cont'd

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$.

Therefore, for $x < 0$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{|x + h| - |x|}{h} \\ &= \lim_{h \rightarrow 0} \frac{-(x + h) - (-x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h} = \lim_{h \rightarrow 0} (-1) = -1 \end{aligned}$$

and so f is differentiable for any $x < 0$.

Example 5 – Solution

cont'd

For $x = 0$ we have to investigate

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|0 + h| - |0|}{h} \quad (\text{if it exists}) \end{aligned}$$

Let's compute the left and right limits separately:

$$\lim_{h \rightarrow 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{|h|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} 1 = 1$$

and

$$\lim_{h \rightarrow 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = \lim_{h \rightarrow 0^-} (-1) = -1$$

Example 5 – Solution

cont'd

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 5(b).

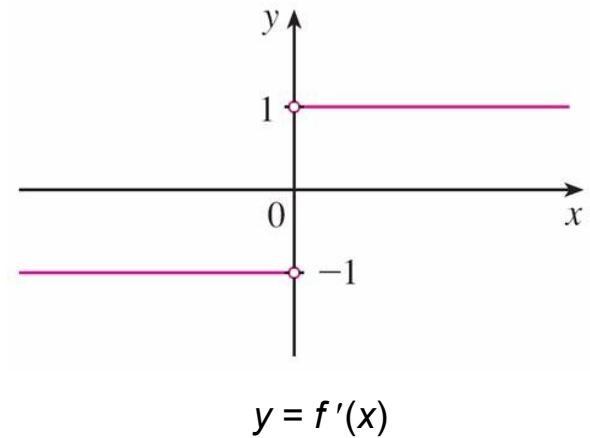
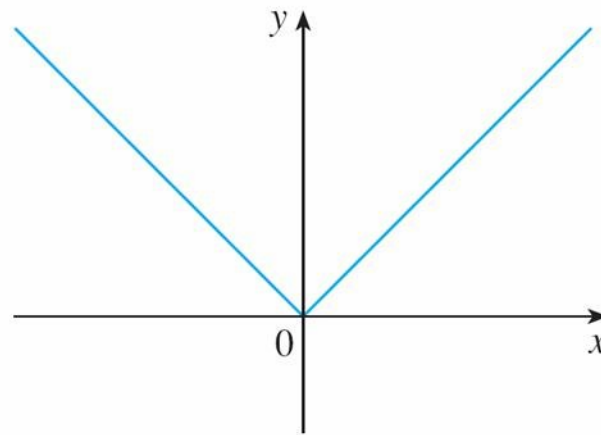


Figure 5(b)

Example 5 – Solution

cont'd

The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at $(0, 0)$. [See Figure 5(a).]



$$y = f(x) = |x|$$


Figure 5(a)

Other Notations

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

4 Theorem If f is differentiable at a , then f is continuous at a .

Note: The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable.

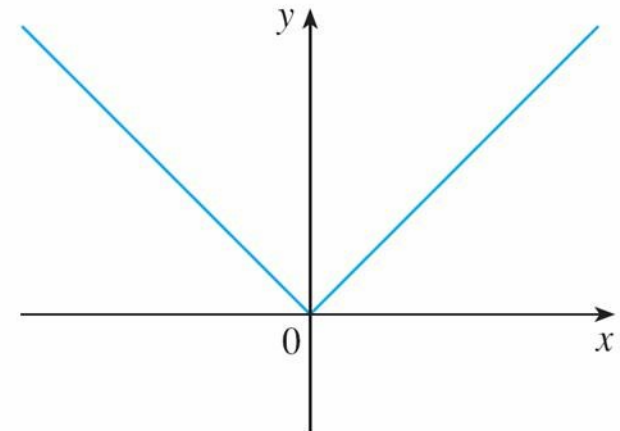


How Can a Function Fail to Be Differentiable?

How Can a Function Fail to Be Differentiable?

We saw that the function $y = |x|$ in Example 5 is not differentiable at 0 and Figure 5(a) shows that its graph changes direction abruptly when $x = 0$.

In general, if the graph of a function f has a “corner” or “kink” in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]



$$y = f(x) = |x|$$

| Figure 5(a)

How Can a Function Fail to Be Differentiable?

Theorem 4 gives another way for a function not to have a derivative. It says that if f is not continuous at a , then f is not differentiable at a . So at any discontinuity (for instance, a jump discontinuity) f fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, f is continuous at a and

$$\lim_{x \rightarrow a} |f'(x)| = \infty$$

How Can a Function Fail to Be Differentiable?

This means that the tangent lines become steeper and steeper as $x \rightarrow a$. Figure 6 shows one way that this can happen; Figure 7(c) shows another.

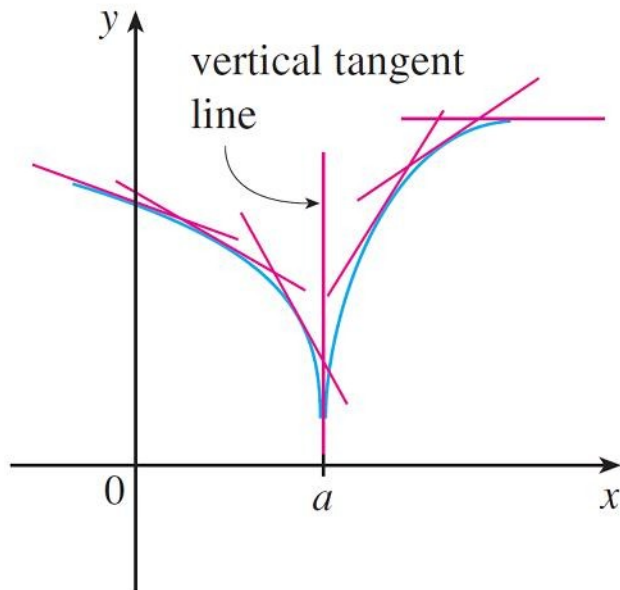
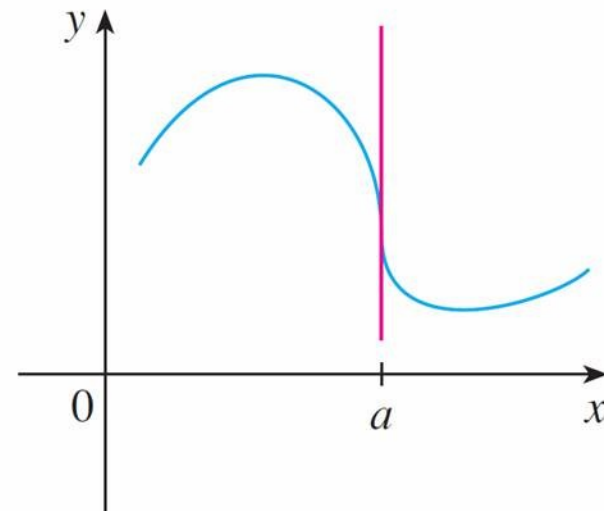


Figure 6

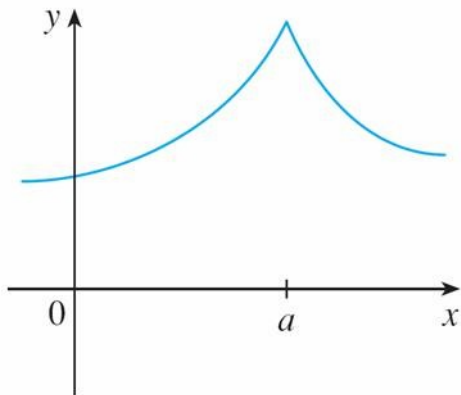


A vertical tangent

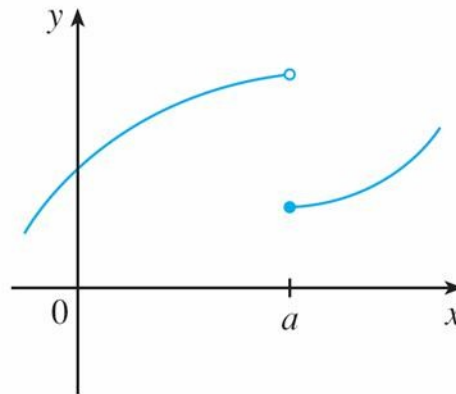
Figure 7(c)

How Can a Function Fail to Be Differentiable?

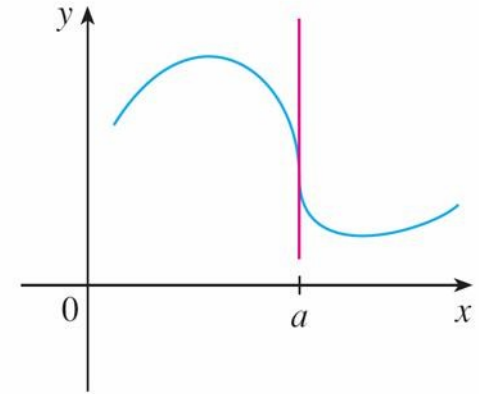
Figure 7 illustrates the three possibilities that we have discussed.



(a) A corner



(b) A discontinuity



(c) A vertical tangent

Three ways for f not to be differentiable at a

Figure 7



Higher Derivatives

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f .

Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2}$$

Example 6

If $f(x) = x^3 - x$, find and interpret $f''(x)$.

Solution:

The first derivative of $f(x) = x^3 - x$ is $f'(x) = 3x^2 - 1$.

So the second derivative is

$$\begin{aligned} f''(x) &= (f')'(x) \\ &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \end{aligned}$$

Example 6 – Solution

cont'd

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h) \\ &= 6x \end{aligned}$$

The graphs of f , f' , and f'' are shown in Figure 10.

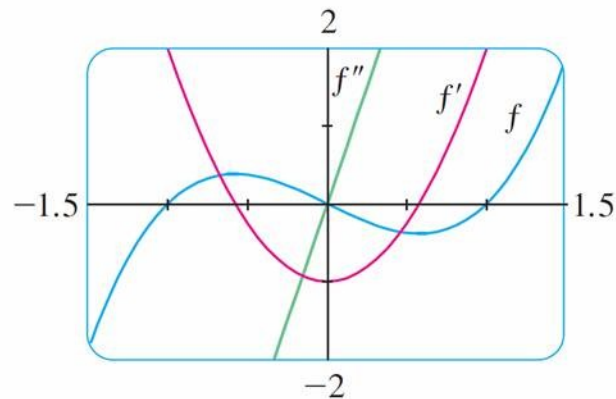


Figure 10

Example 6 – *Solution*

cont'd

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 10 that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope. So the graphs serve as a check on our calculations.

Higher Derivatives

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration*, which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

Higher Derivatives

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Higher Derivatives

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$.

If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx} \left(\frac{d^2 y}{dx^2} \right) = \frac{d^3 y}{dx^3}$$

Higher Derivatives

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$.

In general, the n th derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times.

If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Higher Derivatives

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line.

Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

Higher Derivatives

Thus the jerk j is the rate of change of acceleration.

It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.