

3

Differentiation Rules



3.1

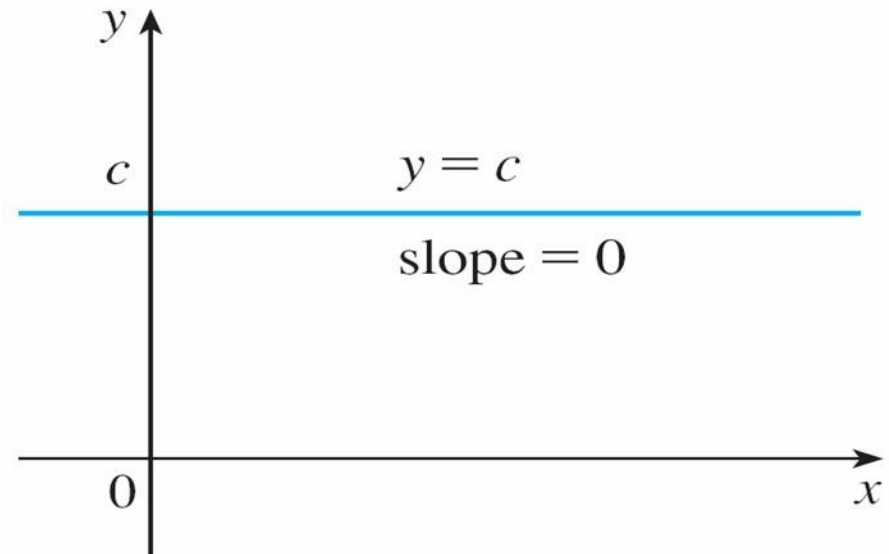
Derivatives of Polynomials and Exponential Functions

Derivatives of Polynomials and Exponential Functions

In this section we learn how to differentiate constant functions, power functions, polynomials, and exponential functions.

Let's start with the simplest of all functions, the constant function $f(x) = c$.

The graph of this function is the horizontal line $y = c$, which has slope 0, so we must have $f'(x) = 0$.
(See Figure 1.)



The graph of $f(x) = c$ is the line $y = c$, so $f'(x) = 0$.

Figure 1

Derivatives of Polynomials and Exponential Functions

A formal proof, from the definition of a derivative, is also easy:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0$$

In Leibniz notation, we write this rule as follows.

Derivative of a Constant Function

$$\frac{d}{dx} (c) = 0$$

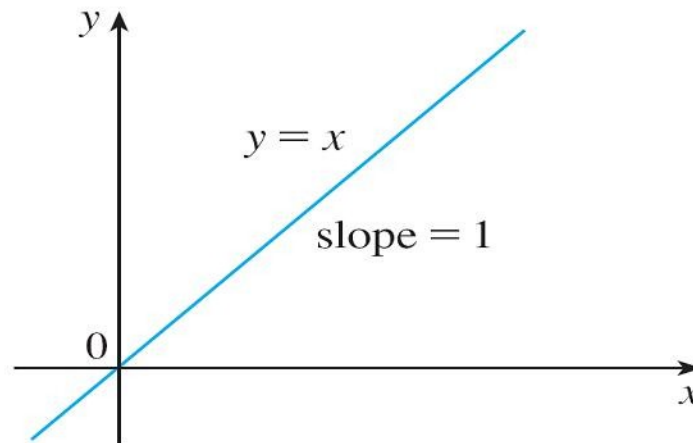


Power Functions

Power Functions

We next look at the functions $f(x) = x^n$, where n is a positive integer.

If $n = 1$, the graph of $f(x) = x$ is the line $y = x$, which has slope 1. (See Figure 2.)



The graph of $f(x) = x$ is the line $y = x$, so $f'(x) = 1$.

Figure 2

Power Functions

So

1

$$\frac{d}{dx}(x) = 1$$

(You can also verify Equation 1 from the definition of a derivative.)

We have already investigated the cases $n = 2$ and $n = 3$. We found that

2

$$\frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(x^3) = 3x^2$$

Power Functions

For $n = 4$ we find the derivative of $f(x) = x^4$ as follows:

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h} \\&= \lim_{h \rightarrow 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\&= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\&= \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3) \\&= 4x^3\end{aligned}$$

Power Functions

Thus

$$\boxed{3} \quad \frac{d}{dx} (x^4) = 4x^3$$

Comparing the equations in $\boxed{1}$, $\boxed{2}$, and $\boxed{3}$, we see a pattern emerging.

It seems to be a reasonable guess that, when n is a positive integer, $(d/dx)(x^n) = nx^{n-1}$. This turns out to be true.

The Power Rule If n is a positive integer, then

$$\frac{d}{dx} (x^n) = nx^{n-1}$$

Example 1

(a) If $f(x) = x^6$, then $f'(x) = 6x^5$.

(b) If $y = x^{1000}$, then $y' = 1000x^{999}$.

(c) If $y = t^4$, then $\frac{dy}{dt} = 4t^3$.

(d) $\frac{d}{dr} (r^3) = 3r^2$

Power Functions

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Power Rule enables us to find tangent lines without having to resort to the definition of a derivative. It also enables us to find *normal lines*.

The **normal line** to a curve C at a point P is the line through P that is perpendicular to the tangent line at P .



New Derivatives from Old

New Derivatives from Old

When new functions are formed from old functions by addition, subtraction, or multiplication by a constant, their derivatives can be calculated in terms of derivatives of the old functions.

In particular, the following formula says that *the derivative of a constant times a function is the constant times the derivative of the function.*

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx} [cf(x)] = c \frac{d}{dx} f(x)$$

Example 4

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx} (3x^4) &= 3 \frac{d}{dx} (x^4) \\ &= 3(4x^3) \\ &= 12x^3 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{d}{dx} (-x) &= \frac{d}{dx} [(-1)x] \\ &= (-1) \frac{d}{dx} (x) \\ &= -1(1) \\ &= -1 \end{aligned}$$

New Derivatives from Old

The next rule tells us that *the derivative of a sum of functions is the sum of the derivatives.*

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(f + g + h)' = [(f + g) + h]' = (f + g)' + h' = f' + g' + h'$$

New Derivatives from Old

By writing $f - g$ as $f + (-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get the following formula.

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx} [f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial, as the following examples demonstrate.



Exponential Functions

Exponential Functions

Let's try to compute the derivative of the exponential function $f(x) = a^x$ using the definition of a derivative:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x a^h - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x (a^h - 1)}{h} \end{aligned}$$

The factor a^x doesn't depend on h , so we can take it in front of the limit:

$$f'(x) = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

Exponential Functions

Notice that the limit is the value of the derivative of f at 0, that is,

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = f'(0)$$

Therefore we have shown that if the exponential function $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere and

$$\boxed{4} \quad f'(x) = f'(0) a^x$$

This equation says that *the rate of change of any exponential function is proportional to the function itself.* (The slope is proportional to the height.)

Exponential Functions

Numerical evidence for the existence of $f'(0)$ is given in the table at the right for the cases $a = 2$ and $a = 3$.

(Values are stated correct to four decimal places.)

It appears that the limits exist and

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

$$\text{for } a = 2, \quad f'(0) = \lim_{h \rightarrow 0} \frac{2^h - 1}{h} \approx 0.69$$

$$\text{for } a = 3, \quad f'(0) = \lim_{h \rightarrow 0} \frac{3^h - 1}{h} \approx 1.10$$

Exponential Functions

In fact, it can be proved that these limits exist and, correct to six decimal places, the values are

$$\left. \frac{d}{dx} (2^x) \right|_{x=0} \approx 0.693147 \qquad \left. \frac{d}{dx} (3^x) \right|_{x=0} \approx 1.098612$$

Thus, from Equation 4, we have

$$\boxed{5} \quad \frac{d}{dx} (2^x) \approx (0.69)2^x \qquad \frac{d}{dx} (3^x) \approx (1.10)3^x$$

Of all possible choices for the base a in Equation 4, the simplest differentiation formula occurs when $f'(0) = 1$.

Exponential Functions

In view of the estimates of $f'(0)$ for $a = 2$ and $a = 3$, it seems reasonable that there is a number a between 2 and 3 for which $f'(0) = 1$.

It is traditional to denote this value by the letter e . Thus we have the following definition.

Definition of the Number e

e is the number such that
$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

Exponential Functions

Geometrically, this means that of all the possible exponential functions $y = a^x$, the function $f(x) = e^x$ is the one whose tangent line at $(0, 1)$ has a slope $f'(0)$ that is exactly 1. (See Figures 6 and 7.)

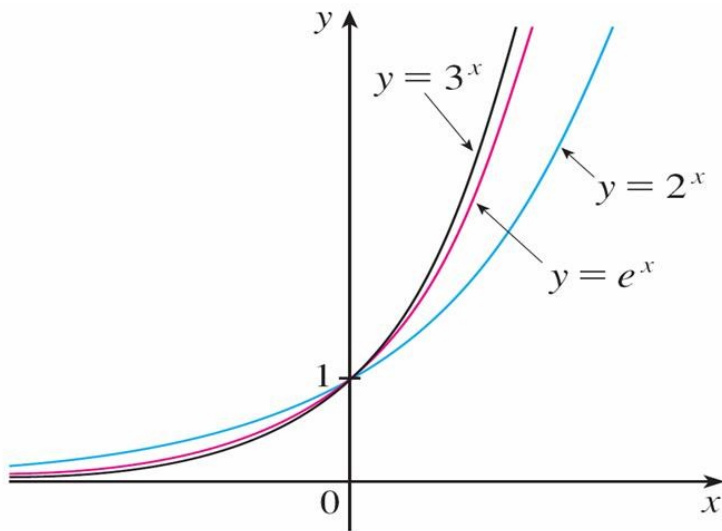


Figure 6

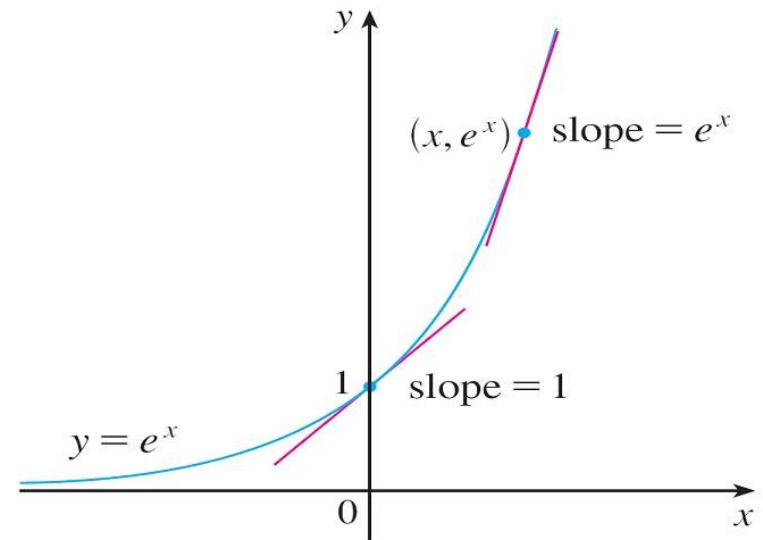


Figure 7

Exponential Functions

If we put $a = e$ and, therefore, $f'(0) = 1$ in Equation 4, it becomes the following important differentiation formula.

Derivative of the Natural Exponential Function

$$\frac{d}{dx} (e^x) = e^x$$

Thus the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ is equal to the y -coordinate of the point (see Figure 7).

Example 8

If $f(x) = e^x - x$, find f' and f'' . Compare the graphs of f and f' .

Solution:

Using the Difference Rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx} (e^x - x) \\ &= \frac{d}{dx} (e^x) - \frac{d}{dx} (x) \\ &= e^x - 1 \end{aligned}$$

Example 8 – *Solution*

cont'd

We defined the second derivative as the derivative of f' , so

$$\begin{aligned} f''(x) &= \frac{d}{dx} (e^x - 1) \\ &= \frac{d}{dx} (e^x) - \frac{d}{dx} (1) \\ &= e^x \end{aligned}$$

Example 8 – Solution

cont'd

The function f and its derivative f' are graphed in Figure 8.

Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing.

When $x < 0$, $f'(x)$ is negative and f is decreasing.

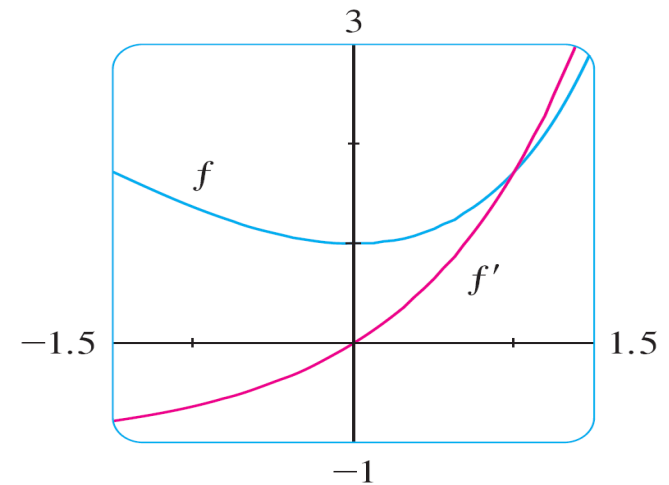


Figure 8