

3

Differentiation Rules



3.2

The Product and Quotient Rules



The Product Rule

The Product Rule

By analogy with the Sum and Difference Rules, one might be tempted to guess, that the derivative of a product is the product of the derivatives.

We can see, however, that this guess is wrong by looking at a particular example.

Let $f(x) = x$ and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$.

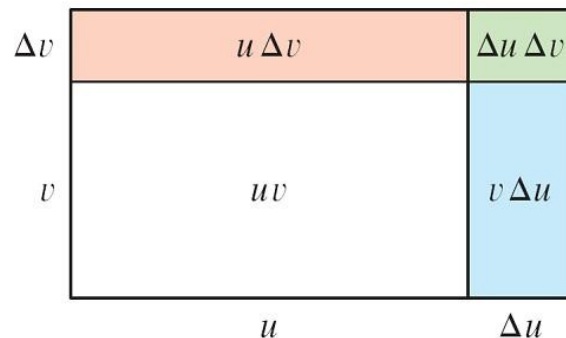
But $(fg)(x) = x^3$, so $(fg)'(x) = 3x^2$. Thus $(fg)' \neq f'g'$.

The Product Rule

The correct formula was discovered by Leibniz and is called the Product Rule.

Before stating the Product Rule, let's see how we might discover it.

We start by assuming that $u = f(x)$ and $v = g(x)$ are both positive differentiable functions. Then we can interpret the product uv as an area of a rectangle (see Figure 1).



The geometry of the Product Rule
Figure 1

The Product Rule

If x changes by an amount Δx , then the corresponding changes in u and v are

$$\Delta u = f(x + \Delta x) - f(x) \quad \Delta v = g(x + \Delta x) - g(x)$$

and the new value of the product, $(u + \Delta u)(v + \Delta v)$, can be interpreted as the area of the large rectangle in Figure 1 (provided that Δu and Δv happen to be positive).

The change in the area of the rectangle is

$$\begin{aligned} \boxed{1} \quad \Delta(uv) &= (u + \Delta u)(v + \Delta v) - uv = u \Delta v + v \Delta u + \Delta u \Delta v \\ &= \text{the sum of the three shaded areas} \end{aligned}$$

The Product Rule

If we divide by Δx , we get

$$\frac{\Delta(uv)}{\Delta x} = u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}$$

If we now let $\Delta x \rightarrow 0$, we get the derivative of uv :

$$\begin{aligned} \frac{d}{dx}(uv) &= \lim_{\Delta x \rightarrow 0} \frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left(u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x} \right) \\ &= u \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \rightarrow 0} \Delta u \right) \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} \right) \\ &= u \frac{dv}{dx} + v \frac{du}{dx} + 0 \cdot \frac{dv}{dx} \end{aligned}$$

The Product Rule

$$\boxed{2} \quad \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

(Notice that $\Delta u \rightarrow 0$ as $\Delta x \rightarrow 0$ since f is differentiable and therefore continuous.)

Although we started by assuming (for the geometric interpretation) that all the quantities are positive, we notice that Equation 1 is always true. (The algebra is valid whether u , v , Δu , Δv and are positive or negative.)

The Product Rule

So we have proved Equation 2, known as the Product Rule, for all differentiable functions u and v .

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx} [f(x)g(x)] = f(x) \frac{d}{dx} [g(x)] + g(x) \frac{d}{dx} [f(x)]$$

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

Example 1

- (a) If $f(x) = xe^x$, find $f'(x)$.
(b) Find the n th derivative, $f^{(n)}(x)$.

Solution:

- (a) By the Product Rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx} (xe^x) \\ &= x \frac{d}{dx} (e^x) + e^x \frac{d}{dx} (x) \\ &= xe^x + e^x \cdot 1 = (x + 1)e^x \end{aligned}$$

Example 1 – Solution

cont'd

(b) Using the Product Rule a second time, we get

$$\begin{aligned} f''(x) &= \frac{d}{dx} [(x + 1)e^x] \\ &= (x + 1) \frac{d}{dx} (e^x) + e^x \frac{d}{dx} (x + 1) \\ &= (x + 1)e^x + e^x \cdot 1 \\ &= (x + 2)e^x \end{aligned}$$

Example 1 – *Solution*

cont'd

Further applications of the Product Rule give

$$f'''(x) = (x + 3)e^x \quad f^{(4)}(x) = (x + 4)e^x$$

In fact, each successive differentiation adds another term e^x , so

$$f^{(n)}(x) = (x + n)e^x$$



The Quotient Rule

The Quotient Rule

We find a rule for differentiating the quotient of two differentiable functions $u = f(x)$ and $v = g(x)$ in much the same way that we found the Product Rule.

If x , u , and v change by amounts Δx , Δu , and Δv , then the corresponding change in the quotient u/v is

$$\begin{aligned}\Delta\left(\frac{u}{v}\right) &= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{(u + \Delta u)v - u(v + \Delta v)}{v(v + \Delta v)} \\ &= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}\end{aligned}$$

The Quotient Rule

so

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{\Delta x \rightarrow 0} \frac{\Delta(u/v)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{v \frac{\Delta u}{\Delta x} - u \frac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

As $\Delta x \rightarrow 0$, $\Delta v \rightarrow 0$ also, because $v = g(x)$ is differentiable and therefore continuous.

Thus, using the Limit Laws, we get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} - u \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x}}{v \lim_{\Delta x \rightarrow 0} (v + \Delta v)} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

The Quotient Rule

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Example 4

Let $y = \frac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6) \frac{d}{dx} (x^2 + x - 2) - (x^2 + x - 2) \frac{d}{dx} (x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

The Quotient Rule

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf'$$

$$(f + g)' = f' + g'$$

$$(f - g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$