# 3

# Derivatives

# 3.5 Implicit Differentiation

# Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \quad \text{or} \quad y = x \sin x$$

or, in general, $y = f(x)$.

Some functions, however, are defined implicitly by a relation between $x$ and $y$ such as

**1** $$x^2 + y^2 = 25$$

or

**2** $$x^3 + y^3 = 6xy$$

# Implicit Differentiation

In some cases it is possible to solve such an equation for *y* as an explicit function (or several functions) of *x*.

For instance, if we solve Equation 1 for *y*, we get $y = \pm\sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$.

# Implicit Differentiation

The graphs of $f$ and $g$ are the upper and lower semicircles of the circle $x^2 + y^2 = 25$. (See Figure 1.)

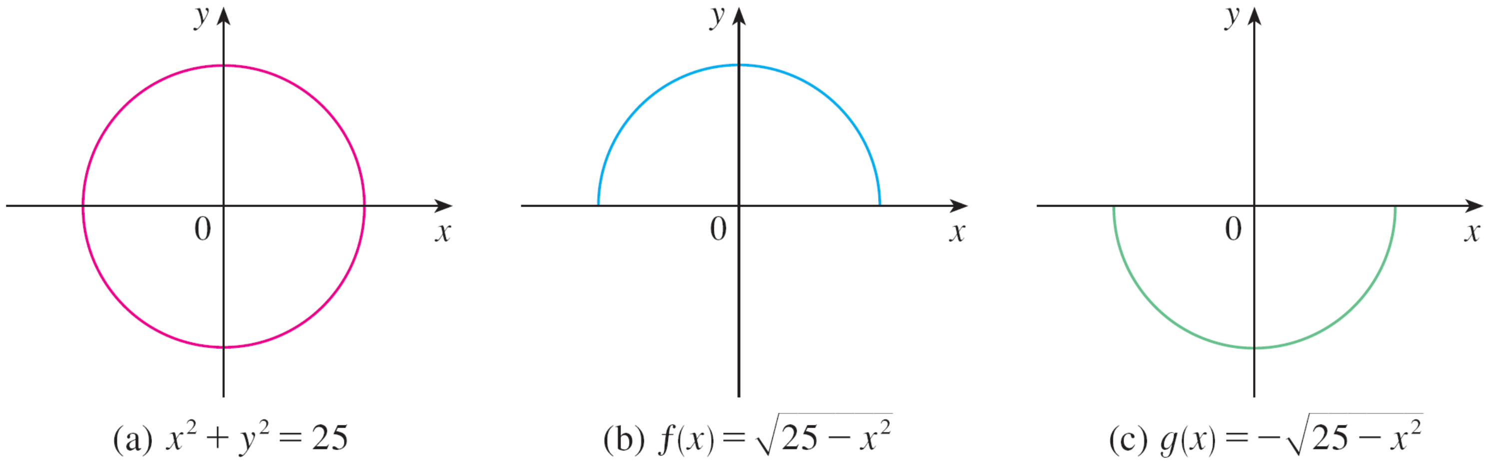
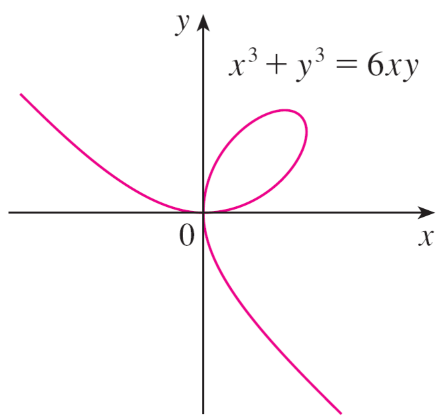



**Figure 1**

# Implicit Differentiation

It's not easy to solve Equation 2 for *y* explicitly as a function of *x* by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated.)

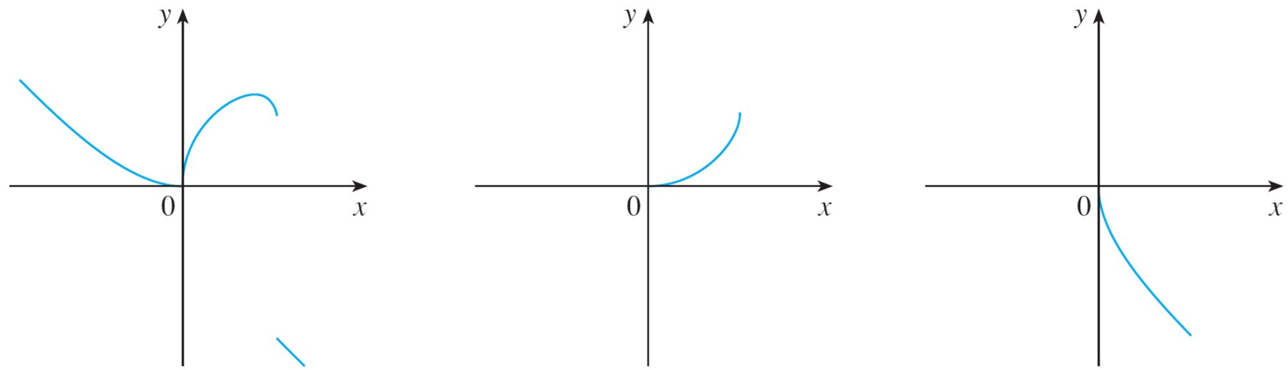
Nonetheless, [2] is the equation of a curve called the **folium of Descartes** shown in Figure 2 and it implicitly defines *y* as several functions of *x*.




The folium of Descartes **Figure 2**

# Implicit Differentiation

The graphs of three such functions are shown in Figure 3.




Graphs of three functions defined by the folium of Descartes

**Figure 3**

When we say that is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)^3] = 6xf(x)$$

is true for all values of in the domain of .

# Implicit Differentiation

Fortunately, we don't need to solve an equation for $y$ in terms of $x$ in order to find the derivative of $y$. Instead we can use the method of **implicit differentiation**.

This consists of differentiating both sides of the equation with respect to $x$ and then solving the resulting equation for $y'$.

In the examples and exercises of this section it is always assumed that the given equation determines $y$ implicitly as a differentiable function of $x$ so that the method of implicit differentiation can be applied.

# Example 1

**(a)** If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

**(b)** Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

Solution 1:

**(a)** Differentiate both sides of the equation $x^2 + y^2 = 25$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

# Example 1 – *Solution*

cont'd

Remembering that *y* is a function of *x* and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx}$$

$$= 2y\frac{dy}{dx}$$

Thus

$$2x + 2y\frac{dy}{dx} = 0$$

Now we solve this equation for *dy*/*dx*:

$$\frac{dy}{dx} = -\frac{x}{y}$$

# Example 1 – *Solution*

cont'd

**(b)** At the point (3, 4) we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at (3, 4) is therefore

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

Solution 2:

**(b)** Solving the equation $x^2 + y^2 = 25$, we get $y = \pm\sqrt{25 - x^2}$.

$\sqrt{25 - x^2}$

The point (3, 4) lies on the upper semi $\sqrt{25 - x^2}$

and so we consider the function $f(x) =$ .

# Example 1 – *Solution*

cont'd

Differentiating *f* using the Chain Rule, we have

$$f'(x) = \tfrac{1}{2}(25 - x^2)^{-1/2} \frac{d}{dx}(25 - x^2)$$

$$= \tfrac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}}$$

So

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

and, as in Solution 1, an equation of the tangent is $3x + 4y = 25$.