3 Differentiation Rules

3.6 Derivatives of Logarithmic Functions

Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic functions $y = \log_a x$ and, in particular, the natural logarithmic function $y = \ln x$. [It can be proved that logarithmic functions are differentiable; this is certainly plausible from their graphs (see Figure 12 in Section 1.6).]

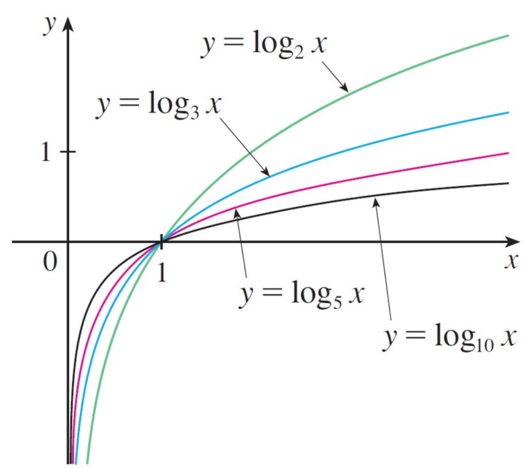

Figure 12

Derivatives of Logarithmic Functions

1

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

In general, if we combine Formula 2 with the Chain Rule, we get

3

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

or

$$\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

Example 2

Find $\frac{d}{dx} \ln(\sin x)$.

Solution:

Using [3], we have

$$\frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \frac{d}{dx} (\sin x) = \frac{1}{\sin x} \cos x$$

$$= \cot x$$

4 $$\frac{d}{dx} (\ln |x|) = \frac{1}{x}$$

Logarithmic Differentiation

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms.

The method used in the next example is called **logarithmic differentiation**.

Example 15

Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}$.

Solution:

We take logarithms of both sides of the equation and use the properties of logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2 + 1) - 5\ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2 + 1} - 5\cdot\frac{3}{3x + 2}$$

Example 15 – *Solution*

cont'd

Solving for *dy*/*dx*, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Because we have an explicit expression for *y*, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Logarithmic Differentiation

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

The Number *e* as a Limit

The Number e as a Limit

If $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1 + x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1 + x)$$

$$= \lim_{x \to 0} \ln(1 + x)^{1/x}$$

The Number *e* as a Limit

Because $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1 + x)^{1/x} = 1$$

Then, by the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}}$$

$$= \lim_{x \to 0} (1 + x)^{1/x}$$

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$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

The Number *e* as a Limit

Formula 5 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ in Figure 4 and a table of values for small values of x. This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$

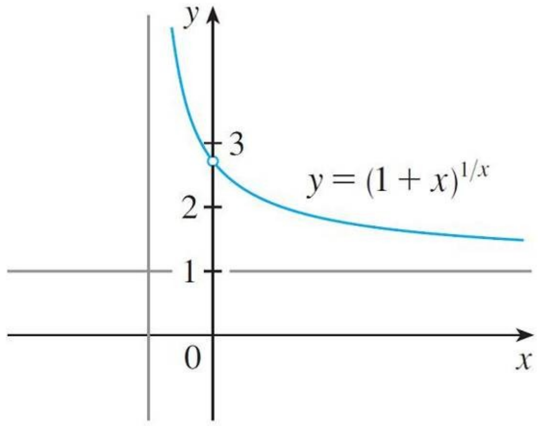

Figure 4

x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

The Number e as a Limit

If we put $n = 1/x$ in Formula 5, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

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$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$