

# 3

## Derivatives



## 3.7

## Rates of Change in the Natural and Social Sciences

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# Rates of Change in the Natural and Social Sciences

We know that if  $y = f(x)$ , then the derivative  $dy/dx$  can be interpreted as the rate of change of  $y$  with respect to  $x$ . In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 2.6 the basic idea behind rates of change. If  $x$  changes from  $x_1$  to  $x_2$ , then the change in  $x$  is

$$\Delta x = x_2 - x_1$$

and the corresponding change in  $y$  is

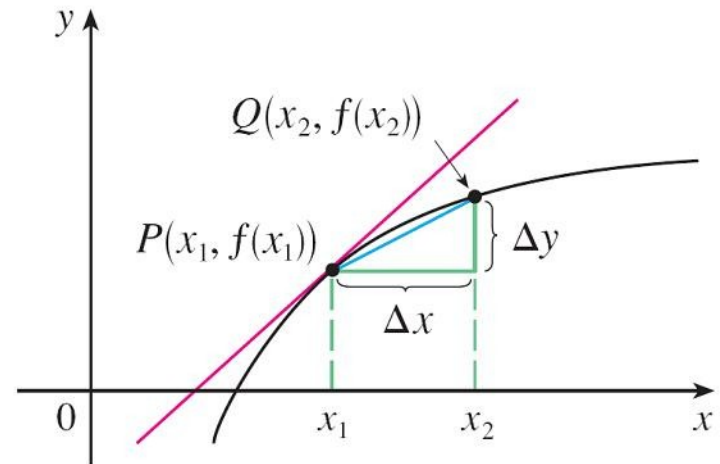
$$\Delta y = f(x_2) - f(x_1)$$

# Rates of Change in the Natural and Social Sciences

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change of  $y$  with respect to  $x$**  over the interval  $[x_1, x_2]$  and can be interpreted as the slope of the secant line  $PQ$  in Figure 1.



$m_{PQ}$  = average rate of change  
 $m = f'(x_1)$  = instantaneous rate of change

Figure 1

# Rates of Change in the Natural and Social Sciences

Its limit as  $\Delta x \rightarrow 0$  is the derivative  $f'(x_1)$ , which can therefore be interpreted as the **instantaneous rate of change of  $y$  with respect to  $x$**  or the slope of the tangent line at  $P(x_1, f(x_1))$ .

Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$



# Physics

# Physics

If  $s = f(t)$  is the position function of a particle that is moving in a straight line, then  $\Delta s/\Delta t$  represents the average velocity over a time period  $\Delta t$ , and  $v = ds/dt$  represents the instantaneous **velocity** (the rate of change of displacement with respect to time).

The instantaneous rate of change of velocity with respect to time is **acceleration**:  $a(t) = v'(t) = s''(t)$ .

Now that we know the differentiation formulas, we are able to solve problems involving the motion of objects more easily.

# Example 1

The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where  $t$  is measured in seconds and  $s$  in meters.

- (a) Find the velocity at time  $t$ .
- (b) What is the velocity after 2 s? After 4 s?
- (c) When is the particle at rest?
- (d) When is the particle moving forward (that is, in the positive direction)?

# Example 1

cont'd

- (e) Draw a diagram to represent the motion of the particle.
- (f) Find the total distance traveled by the particle during the first five seconds.
- (g) Find the acceleration at time  $t$  and after 4 s.
- (h) Graph the position, velocity, and acceleration functions for  $0 \leq t \leq 5$ .
- (i) When is the particle speeding up? When is it slowing down?

# Example 1 – *Solution*

(a) The velocity function is the derivative of the position function.

$$s = f(t) = t^3 - 6t^2 + 9t$$

$$v(t) \frac{ds}{dt} = 3t^2 - 12t + 9$$

# Example 1 – Solution

cont'd

(b) The velocity after 2 s means the instantaneous velocity when  $t = 2$ , that is,

$$\begin{aligned}v(2) \frac{ds}{dt} \Big|_{t=2} &= 3(2)^2 - 12(2) + 9 \\ &= -3 \text{ m/s}\end{aligned}$$

The velocity after 4 s is

$$\begin{aligned}v(4) &= 3(4)^2 - 12(4) + 9 \\ &= 9 \text{ m/s}\end{aligned}$$

# Example 1 – *Solution*

cont'd

(c) The particle is at rest when  $v(t) = 0$ , that is,

$$\begin{aligned}3t^2 - 12t + 9 &= 3(t^2 - 4t + 3) \\ &= 3(t - 1)(t - 3) \\ &= 0\end{aligned}$$

and this is true when  $t = 1$  or  $t = 3$ .

Thus the particle is at rest after 1 s and after 3 s.

# Example 1 – *Solution*

cont'd

(d) The particle moves in the positive direction when

$v(t) > 0$ , that is,

$$3t^2 - 12t + 9 = 3(t - 1)(t - 3) > 0$$

This inequality is true when both factors are positive ( $t > 3$ ) or when both factors are negative ( $t < 1$ ).

Thus the particle moves in the positive direction in the time intervals  $t < 1$  and  $t > 3$ .

It moves backward (in the negative direction) when  $1 < t < 3$ .

# Example 1 – Solution

cont'd

(e) Using the information from part (d) we make a schematic sketch in Figure 2 of the motion of the particle back and forth along a line (the  $s$ -axis).

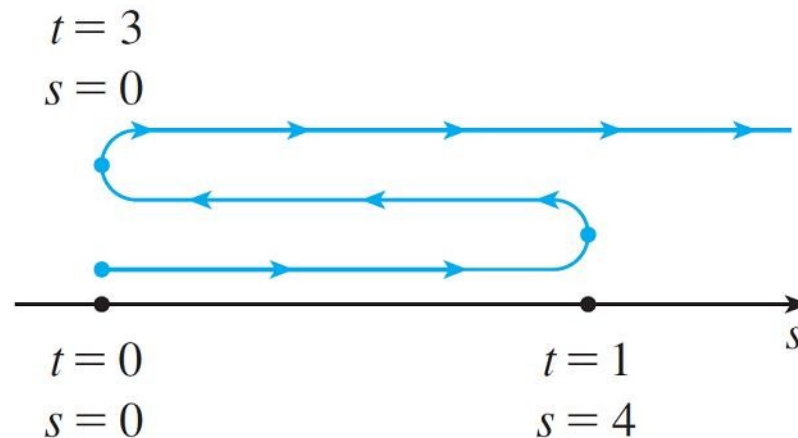


Figure 2

# Example 1 – *Solution*

cont'd

(f) Because of what we learned in parts (d) and (e), we need to calculate the distances traveled during the time intervals  $[0, 1]$ ,  $[1, 3]$ , and  $[3, 5]$  separately.

The distance traveled in the first second is

$$|f(1) - f(0)| = |4 - 0| = 4 \text{ m}$$

From  $t = 1$  to  $t = 3$  the distance traveled is

$$|f(3) - f(1)| = |0 - 4| = 4 \text{ m}$$

From  $t = 3$  to  $t = 5$  the distance traveled is

$$|f(5) - f(3)| = |20 - 0| = 20 \text{ m}$$

The total distance is  $4 + 4 + 20 = 28 \text{ m}$ .

# Example 1 – *Solution*

cont'd

(g) The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt}$$

$$= 6t - 12$$

$$a(4) = 6(4) - 12$$

$$= 12 \text{ m/s}^2$$

# Example 1 – Solution

cont'd

(h) Figure 3 shows the graphs of  $s$ ,  $v$ , and  $a$ .

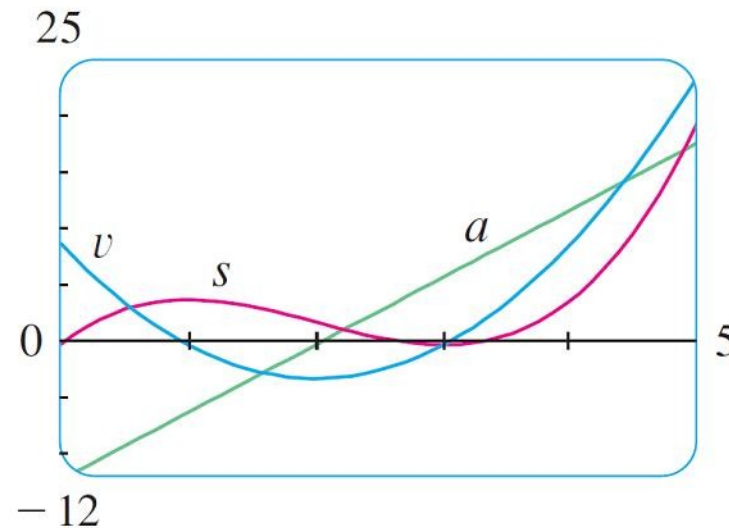


Figure 3

# Example 1 – *Solution*

cont'd

- (i) The particle speeds up when the velocity is positive and increasing ( $v$  and  $a$  are both positive) and also when the velocity is negative and decreasing ( $v$  and  $a$  are both negative).

In other words, the particle speeds up when the velocity and acceleration have the same sign. (The particle is pushed in the same direction it is moving.)

From Figure 3 we see that this happens when  $1 < t < 2$

# Example 1 – Solution

cont'd

The particle slows down when  $v$  and  $a$  have opposite signs, that is, when  $0 \leq t < 1$  and when  $2 < t < 3$ .

Figure 4 summarizes the motion of the particle.

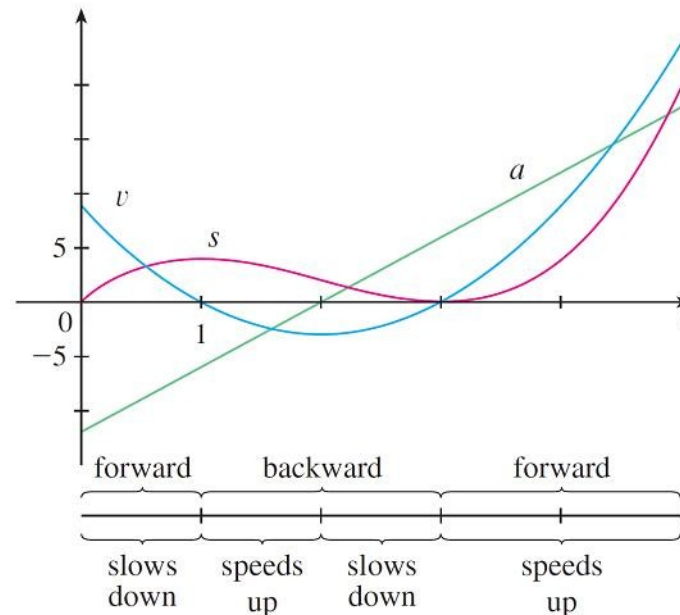


Figure 4

## Example 2

If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ( $\rho = m/l$ ) and measured in kilograms per meter.

Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point  $x$  is  $m = f(x)$ , as shown in Figure 5.

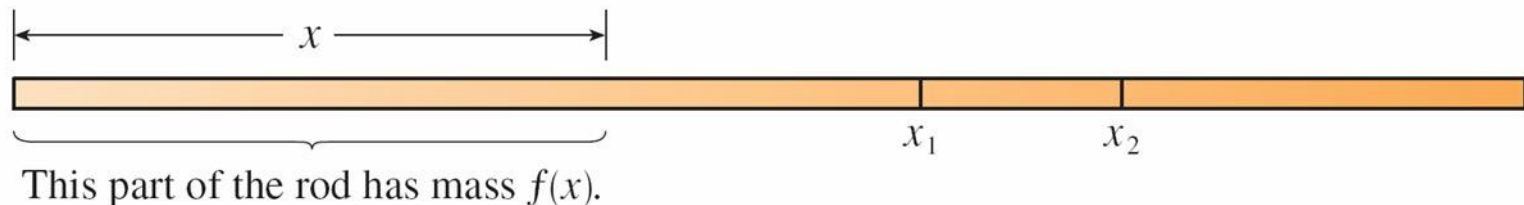


Figure 5

# Example 2

cont'd

The mass of the part of the rod that lies between  $x = x_1$  and  $x = x_2$  is given by  $\Delta m = f(x_2) - f(x_1)$ , so the average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If we now let  $\Delta x \rightarrow 0$  (that is,  $x_2 \rightarrow x_1$ ), we are computing the average density over smaller and smaller intervals.

The **linear density**  $\rho$  at  $x_1$  is the limit of these average densities as  $\Delta x \rightarrow 0$ ; that is, the linear density is the rate of change of mass with respect to length.

# Example 2

cont'd

Symbolically,

$$\rho = \lim_{\Delta x \rightarrow 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Thus the linear density of the rod is the derivative of mass with respect to length.

For instance, if  $m = f(x) = \sqrt{x}$ , where  $x$  is measured in meters and  $m$  in kilograms, then the average density of the part of the rod given by  $1 \leq x \leq 1.2$  is

$$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{\sqrt{1.2} - 1}{0.2}$$

# Example 2

cont'd

$$\approx 0.48 \text{ kgm}$$

while the density right at  $x = 1$  is

$$\begin{aligned}\rho &= \left. \frac{dm}{dx} \right|_{x=1} = \left. \frac{1}{2\sqrt{x}} \right|_{x=1} \\ &= 0.50 \text{ kg/m}\end{aligned}$$

# Example 3

A current exists whenever electric charges move. Figure 6 shows part of a wire and electrons moving through a plane surface, shaded red.

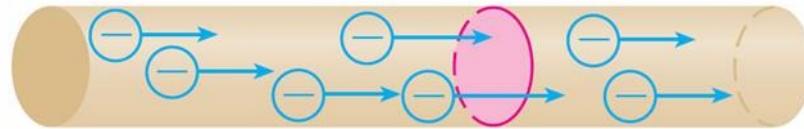


Figure 6

If  $\Delta Q$  is the net charge that passes through this surface during a time period  $\Delta t$ , then the average current during this time interval is defined as

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

# Example 3

cont'd

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the **current  $I$**  at a given time  $t_1$ :

$$I = \lim_{\Delta t \rightarrow 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

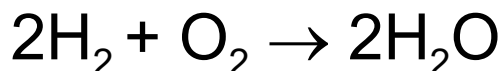
Thus the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes).



# Chemistry

# Example 4

A chemical reaction results in the formation of one or more substances (called *products*) from one or more starting materials (called *reactants*). For instance, the “equation”



indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water.

Let's consider the reaction



where A and B are the reactants and C is the product.

# Example 4

cont'd

The **concentration** of a reactant A is the number of moles (1 mole =  $6.022 \times 10^{23}$  molecules) per liter and is denoted by [A].

The concentration varies during a reaction, so [A], [B], and [C] are all functions of time ( $t$ ).

The average rate of reaction of the product C over a time interval  $t_1 \leq t \leq t_2$  is

$$\frac{\Delta[C]}{\Delta t} = \frac{[C](t_2) - [C](t_1)}{t_2 - t_1}$$

# Example 4

cont'd

But chemists are more interested in the **instantaneous rate of reaction**, which is obtained by taking the limit of the average rate of reaction as the time interval  $\Delta t$  approaches 0:

$$\text{rate of reaction} = \lim_{\Delta t \rightarrow 0} \frac{\Delta[\text{C}]}{\Delta t} = \frac{d[\text{C}]}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative  $d[\text{C}]/dt$  will be positive, and so the rate of reaction of C is positive.

# Example 4

cont'd

The concentrations of the reactants, however, decrease during the reaction, so, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives  $d[A]/dt$  and  $d[B]/dt$ .

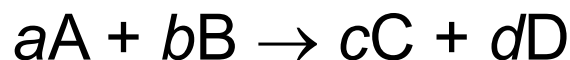
Since [A] and [B] each decrease at the same rate that [C] increases, we have

$$\text{rate of reaction} = \frac{d[C]}{dt} = -\frac{d[A]}{dt} = -\frac{d[B]}{dt}$$

# Example 4

cont'd

More generally, it turns out that for a reaction of the form



we have

$$-\frac{1}{a} \frac{d[A]}{dt} = -\frac{1}{b} \frac{d[B]}{dt} = \frac{1}{c} \frac{d[C]}{dt} = \frac{1}{d} \frac{d[D]}{dt}$$

The rate of reaction can be determined from data and graphical methods. In some cases there are explicit formulas for the concentrations as functions of time, which enable us to compute the rate of reaction.

# Example 5

One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume  $V$  depends on its pressure  $P$ .

We can consider the rate of change of volume with respect to pressure—namely, the derivative  $dV/dP$ . As  $P$  increases,  $V$  decreases, so  $dV/dP < 0$ .

The **compressibility** is defined by introducing a minus sign and dividing this derivative by the volume  $V$ :

$$\text{isothermal compressibility} = \beta = -\frac{1}{V} \frac{dV}{dP}$$

# Example 5

cont'd

Thus  $\beta$  measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume  $V$  (in cubic meters) of a sample of air at 25°C was found to be related to the pressure  $P$  (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

# Example 5

cont'd

The rate of change of  $V$  with respect to  $P$  when  $P = 50$  kPa is

$$\begin{aligned}\frac{dV}{dP} \Big|_{P=50} &= -\frac{5.3}{P^2} \Big|_{P=50} \\ &= -\frac{5.3}{2500} \\ &= -0.00212 \text{ m}^3/\text{kPa}\end{aligned}$$

# Example 5

cont'd

The compressibility at that pressure is

$$\begin{aligned}\beta &= -\frac{1}{V} \frac{dV}{dP} \Big|_{P=50} \\ &= \frac{0.00212}{\frac{5.3}{50}} \\ &= 0.02 \text{ (m}^3\text{/kPa)/m}^3\end{aligned}$$



# Biology

# Example 6

Let  $n = f(t)$  be the number of individuals in an animal or plant population at time  $t$ .

The change in the population size between the times  $t = t_1$  and  $t = t_2$  is  $\Delta n = f(t_2) - f(t_1)$ , and so the average rate of growth during the time period  $t_1 \leq t \leq t_2$  is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period  $\Delta t$  approach 0:

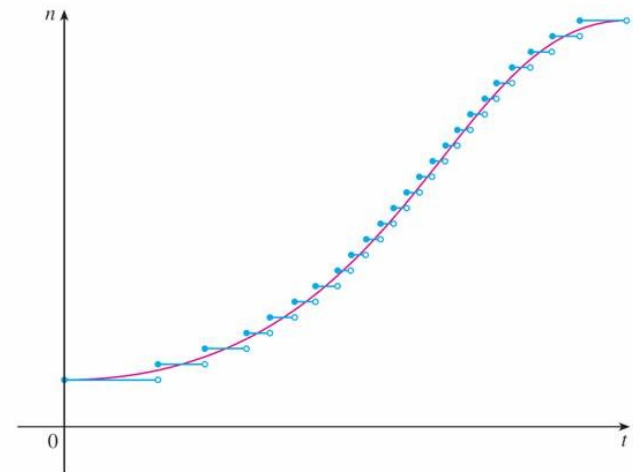
$$\text{growth rate} = \lim_{\Delta t \rightarrow 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

# Example 6

cont'd

Strictly speaking, this is not quite accurate because the actual graph of a population function  $n = f(t)$  would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable.

However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as in Figure 7.



A smooth curve  
approximating  
a growth function  
Figure 7

# Example 6

cont'd

To be more specific, consider a population of bacteria in a homogeneous nutrient medium.

Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour.

If the initial population is  $n_0$  and the time  $t$  is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$

$$f(2) = 2f(1) = 2^2n_0$$

# Example 6

cont'd

$$f(3) = 2f(2) = 2^3 n_0$$

and, in general,

$$f(t) = 2^t n_0$$

The population function is  $n = n_0 2^t$ .

We have shown that

$$\frac{d}{dx} (a^x) = a^x \ln a$$

# Example 6

cont'd

So the rate of growth of the bacteria population at time  $t$  is

$$\begin{aligned}\frac{dn}{dt} &= \frac{d}{dt} (n_0 2^t) \\ &= n_0 2^t \ln 2\end{aligned}$$

For example, suppose that we start with an initial population of  $n_0 = 100$  bacteria. Then the rate of growth after 4 hours is

$$\left. \frac{dn}{dt} \right|_{t=4} = 100 \cdot 2^4 \ln 2$$

# Example 6

cont'd

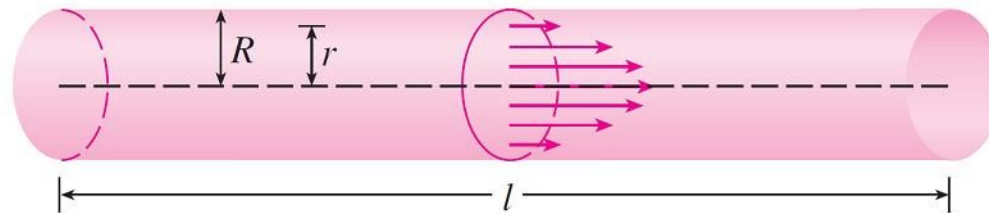
$$= 1600 \ln 2$$

$$\approx 1109$$

This means that, after 4 hours, the bacteria population is growing at a rate of about 1109 bacteria per hour.

# Example 7

When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius  $R$  and length  $l$  as illustrated in Figure 8.



Blood flow in an artery  
Figure 8

Because of friction at the walls of the tube, the velocity  $v$  of the blood is greatest along the central axis of the tube and decreases as the distance  $r$  from the axis increases until  $v$  becomes 0 at the wall.

# Example 7

cont'd

The relationship between  $v$  and  $r$  is given by the **law of laminar flow** discovered by the French physician Jean-Louis-Marie Poiseuille in 1840.

This law states that

1

$$v = \frac{P}{4\eta l} (R^2 - r^2)$$

where  $\eta$  is the viscosity of the blood and  $P$  is the pressure difference between the ends of the tube.

If  $P$  and  $l$  are constant, then  $v$  is a function of  $r$  with domain  $[0, R]$ .

# Example 7

cont'd

The average rate of change of the velocity as we move from  $r = r_1$  outward to  $r = r_2$  is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

and if we let  $\Delta r \rightarrow 0$ , we obtain the **velocity gradient**, that is, the instantaneous rate of change of velocity with respect to  $r$ :

$$\text{velocity gradient} = \lim_{\Delta r \rightarrow 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

# Example 7

cont'd

Using Equation 1, we obtain

$$\begin{aligned}\frac{dv}{dr} &= \frac{P}{4\eta l} (0 - 2r) \\ &= -\frac{Pr}{2\eta l}\end{aligned}$$

For one of the smaller human arteries we can take  $\eta = 0.027$ ,  $R = 0.008$  cm,  $l = 2$  cm, and  $P = 4000$  dynes/cm<sup>2</sup>, which gives

$$v = \frac{4000}{4(0.027)^2} (0.000064 - r^2)$$

# Example 7

cont'd

$$\approx 1.85 \times 10^4(6.4 \times 10^{-5} - r^2)$$

At  $r = 0.02$  cm the blood is flowing at a speed of

$$\begin{aligned}v(0.002) &\approx 1.85 \times 10^4(64 \times 10^{-6} - 4 \times 10^{-6}) \\ &= 1.11 \text{ cm/s}\end{aligned}$$

and the velocity gradient at that point is

$$\left. \frac{dv}{dr} \right|_{r=0.002} = -\frac{4000(0.002)}{2(0.027)^2} \approx -74 \text{ (cm/s)/cm}$$

# Example 7

cont'd

To get a feeling for what this statement means, let's change our units from centimeters to micrometers ( $1 \text{ cm} = 10,000 \text{ } \mu\text{m}$ ). Then the radius of the artery is  $80 \text{ } \mu\text{m}$ .

The velocity at the central axis is  $11,850 \text{ } \mu\text{m/s}$ , which decreases to  $11,110 \text{ } \mu\text{m/s}$  at a distance of  $r = 20 \text{ } \mu\text{m}$ .

The fact that  $dv/dr = -74 \text{ } (\mu\text{m/s})/\mu\text{m}$  means that, when  $r = 20 \text{ } \mu\text{m}$ , the velocity is decreasing at a rate of about  $74 \text{ } \mu\text{m/s}$  for each micrometer that we proceed away from the center.



# Economics

# Example 8

Suppose  $C(x)$  is the total cost that a company incurs in producing  $x$  units of a certain commodity.

The function  $C$  is called a **cost function**. If the number of items produced is increased from  $x_1$  to  $x_2$ , then the additional cost is  $\Delta C = C(x_2) - C(x_1)$ , and the average rate of change of the cost is

$$\begin{aligned}\frac{\Delta C}{\Delta x} &= \frac{C(x_2) - C(x_1)}{x_2 - x_1} \\ &= \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}\end{aligned}$$

# Example 8

cont'd

The limit of this quantity as  $\Delta x \rightarrow 0$ , that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \rightarrow 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since  $x$  often takes on only integer values, it may not make literal sense to let  $\Delta x$  approach 0, but we can always replace  $C(x)$  by a smooth approximating function as in Example 6.]

# Example 8

cont'd

Taking  $\Delta x = 1$  and  $n$  large (so that  $\Delta x$  is small compared to  $n$ ), we have

$$C'(n) \approx C(n + 1) - C(n)$$

Thus the marginal cost of producing  $n$  units is approximately equal to the cost of producing one more unit [the  $(n + 1)$ st unit].

# Example 8

cont'd

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where  $a$  represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to  $x$ , but labor costs might depend partly on higher powers of  $x$  because of overtime costs and inefficiencies involved in large-scale operations.)

# Example 8

cont'd

For instance, suppose a company has estimated that the cost (in dollars) of producing  $x$  items is

$$C(x) = 10,000 + 5x + 0.01x^2$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02(500)$$

$$= \$15/\text{item}$$

# Example 8

cont'd

This gives the rate at which costs are increasing with respect to the production level when  $x = 500$  and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$\begin{aligned} C(501) - C(500) &= [10,000 + 5(501) + 0.01(501)^2] \\ &\quad - [10,000 + 5(500) + 0.01(500)^2] \\ &= \$15.01 \end{aligned}$$

Notice that  $C'(500) \approx C(501) - C(500)$ .



# Other Sciences

# Other Sciences

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks.

An engineer wants to know the rate at which water flows into or out of a reservoir.

An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases.

A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height.



# **A Single Idea, Many Interpretations**

# A Single Idea, Many Interpretations

Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness.

# A Single Idea, Many Interpretations

A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences.

When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences.

This is much more efficient than developing properties of special concepts in each separate science.