3

Derivatives

3.8 Exponential Growth and Decay

Exponential Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individuals in a population of animals or bacteria at time t, then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$; that is, $f'(t) = kf(t)$ for some constant k.

Indeed, under ideal conditions (unlimited environment, adequate nutrition, immunity to disease) the mathematical model given by the equation $f'(t) = kf(t)$ predicts what actually happens fairly accurately.

Exponential Growth and Decay

Another example occurs in nuclear physics where the mass of a radioactive substance decays at a rate proportional to the mass.

In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance.

In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

Exponential Growth and Decay

In general, if *y*(*t*) is the value of a quantity *y* at time *t* and if the rate of change of *y* with respect to *t* is proportional to its size *y*(*t*) at any time, then

1
$$\frac{dy}{dt} = ky$$

where *k* is a constant.

Equation 1 is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). It is called a **differential equation** because it involves an unknown function *y* and its derivative dy/dt.

Exponential Growth and Decay

It's not hard to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself.

Any exponential function of the form $y(t) = Ce^{kt}$, where C is a constant, satisfies

$$y'(t) = C(ke^{kt}) = k(Ce^{kt}) = ky(t)$$

Exponential Growth and Decay

We will see later that *any* function that satisfies $dy/dt = ky$ must be of the form $y = Ce^{kt}$. To see the significance of the constant C, we observe that

$$y(0) = Ce^{k \cdot 0} = C$$

Therefore C is the initial value of the function.

2 Theorem The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

Population Growth

Population Growth

What is the significance of the proportionality constant *k*? In the context of population growth, where *P*(*t*) is the size of a population at time *t*, we can write

$$\boxed{3} \qquad \frac{dP}{dt} = kP \qquad \text{or} \qquad \frac{1}{P}\frac{dP}{dt} = k$$

The quantity

$$\frac{1}{P}\frac{dP}{dt}$$

is the growth rate divided by the population size; it is called the **relative growth rate**.

Population Growth

According to [3], instead of saying "the growth rate is proportional to population size" we could say "the relative growth rate is constant."

Then [2] says that a population with constant relative growth rate must grow exponentially.

Notice that the relative growth rate k appears as the coefficient of t in the exponential function Ce^{kt}.

Population Growth

For instance, if

$$\frac{dP}{dt} = 0.02P$$

and t is measured in years, then the relative growth rate is $k = 0.02$ and the population grows at a relative rate of 2% per year.

If the population at time 0 is P_0, then the expression for the population is

$$P(t) = P_0 e^{0.02t}$$

Example 1

Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Solution:

We measure the time t in years and let $t = 0$ in the year 1950.

Example 1 – *Solution*

cont'd

We measure the population $P(t)$ in millions of people. Then $P(0) = 2560$ and $P(10) = 3040$.

Since we are assuming that $dP/dt = kP$, Theorem 2 gives

$$P(t) = P(0)e^{kt} = 2560e^{kt}$$

$$P(10) = 2560e^{10k} = 3040$$

$$k = \frac{1}{10} \ln \frac{3040}{2560} \approx 0.017185$$

Example 1 – *Solution*

The relative growth rate is about 1.7% per year and the model is

$$P(t) = 2560e^{0.017185t}$$

We estimate that the world population in 1993 was

$$P(43) = 2560e^{0.017185(43)} \approx 5360 \text{ million}$$

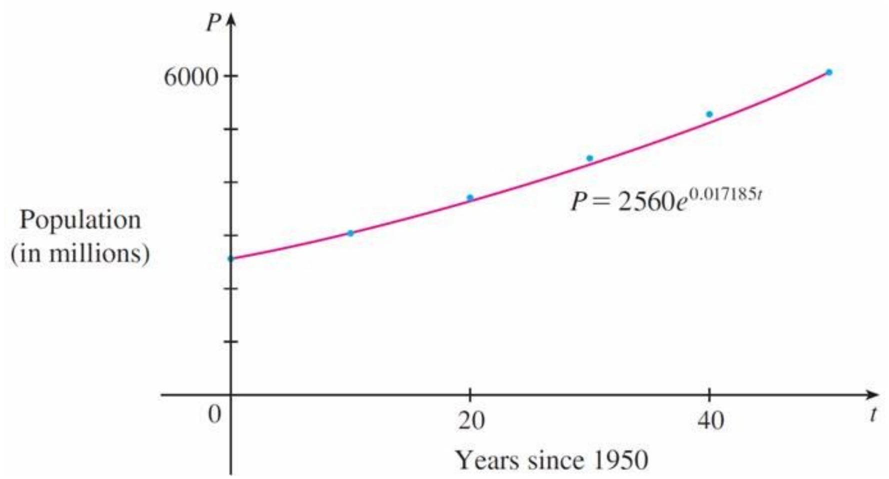
The model predicts that the population in 2020 will be

$$P(70) = 2560e^{0.017185(70)} \approx 8524 \text{ million}$$

Example 1 – *Solution*

cont'd

The graph in Figure 1 shows that the model is fairly accurate to the end of the 20th century (the dots represent the actual population), so the estimate for 1993 is quite reliable. But the prediction for 2020 is riskier.



A model for world population growth in the second half of the 20th century

Figure 1

Radioactive Decay

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate

$$-\frac{1}{m}\frac{dm}{dt}$$

has been found experimentally to be constant. (Since dm/dt is negative, the relative decay rate is positive.) It follows that

$$\frac{dm}{dt} = km$$

where k is a negative constant.

Radioactive Decay

In other words, radioactive substances decay at a rate proportional to the remaining mass. This means that we can use [2] to show that the mass decays exponentially:

$$m(t) = m_0 e^{kt}$$

Physicists express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay.

Example 2

The half-life of radium-226 is 1590 years.

(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t

years.

(b) Find the mass after 1000 years correct to the nearest milligram.

(c) When will the mass be reduced to 30 mg?

Solution:

(a) Let $m(t)$ be the mass of radium-226 (in milligrams) that remains after t years.

Example 2 – *Solution*

cont'd

Then $dm/dt = km$ and $y(0) = 100$, so $\boxed{2}$ gives

$$m(t) = m(0)e^{kt} = 100e^{kt}$$

In order to determine the value of *k*, we use the fact that $y(1590) = \frac{1}{2}(100)$. Thus

$$100e^{1590k} = 50 \qquad \text{so} \qquad e^{1590k} = \frac{1}{2}$$

and

$$1590k = \ln \frac{1}{2} = -\ln 2$$

Example 2 – *Solution*

cont'd

$$k = -\frac{\ln 2}{1590}$$

Therefore

$$m(t) = 100e^{-(\ln 2)t/1590}$$

We could use the fact that $e^{\ln 2} = 2$ to write the expression for $m(t)$ in the alternative form

$$m(t) = 100 \times 2^{-t/1590}$$

Example 2 – *Solution*

cont'd

(b) The mass after 1000 years is

$$m(1000) = 100e^{-(\ln 2)1000/1590} \approx 65 \text{ mg}$$

(c) We want to find the value of t such that $m(t) = 30$, that is,

$$100e^{-(\ln 2)t/1590} = 30 \quad \text{or} \quad e^{-(\ln 2)t/1590} = 0.3$$

We solve this equation for t by taking the natural logarithm of both sides:

$$-\frac{\ln 2}{1590} t = \ln 0.3$$

Example 2 – *Solution*

cont'd

Thus

$$t = -1590 \frac{\ln 0.3}{\ln 2}$$

$$\approx 2762 \text{ years}$$

Radioactive Decay

As a check on our work in Example 2, we use a graphing device to draw the graph of $m(t)$ in Figure 2 together with the horizontal line $m = 30$. These curves intersect when $t \approx 2800$, and this agrees with the answer to part (c).

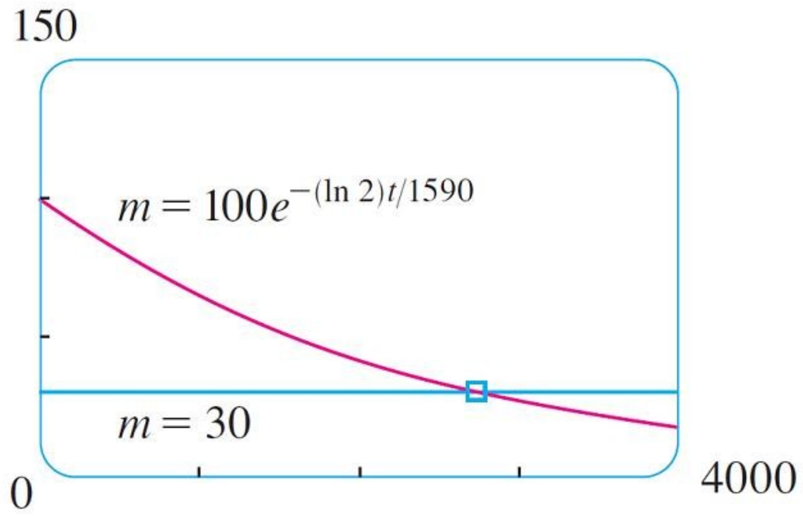


Figure 2

Newton’s Law of Cooling

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming.)

If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant.

Newton's Law of Cooling

This equation is not quite the same as Equation 1, so we make the change of variable $y(t) = T(t) - T_s$. Because T_s is constant, we have $y'(t) = T'(t)$ and so the equation becomes

$$\frac{dy}{dt} = ky$$

We can then use $\boxed{2}$ to find an expression for y, from which we can find T.

Example 3

A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F.

(a) What is the temperature of the soda pop after another half hour?

(b) How long does it take for the soda pop to cool to 50°F?

Solution:

(a) Let $T(t)$ be the temperature of the soda after t minutes.

Example 3 – *Solution*

cont'd

The surrounding temperature is T_s = 44°F, so Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - 44)$$

If we let $y = T - 44$, then $y(0) = T(0) - 44 = 72 - 44 = 28$,

so y satisfies $\frac{dy}{dt} = ky$ $y(0) = 28$

and by $\boxed{2}$ we have

$$y(t) = y(0)e^{kt} = 28e^{kt}$$

Example 3 – *Solution*

cont'd

We are given that $T(30) = 61$, so $y(30) = 61 – 44 =$
17 and

$$28e^{30}\, e^{30k} = \frac{17}{28}$$

Taking logarithms, we have

$$k = \frac{\ln\left(\frac{17}{28}\right)}{30}$$

$$\approx -0.01663$$

Example 3 – *Solution*

cont'd

Thus

$$y(t) = 28e^{-0.01663t}$$

$$T(t) = 44 + 28e^{-0.01663t}$$

$$T(60) = 44 + 28e^{-0.01663(60)}$$

$$\approx 54.3$$

So after another half hour the pop has cooled to about 54° F.

Example 3 – *Solution*

cont'd

(b) We have $T(t) = 50$ when

$$44 + 28e^{-0.01663t} = 50$$

$$e^{-0.01663t} = \frac{6}{28}$$

$$t = \frac{\ln\left(\frac{6}{28}\right)}{-0.01663}$$

$$\approx 92.6$$

The pop cools to 50°F after about 1 hour 33 minutes.

Newton's Law of Cooling

Notice that in Example 3, we have

$$\lim_{t \to \infty} T(t) = \lim_{t \to \infty} (44 + 28e^{-0.01663t}) = 44 + 28 \cdot 0 = 44$$

which is to be expected. The graph of the temperature function is shown in Figure 3.

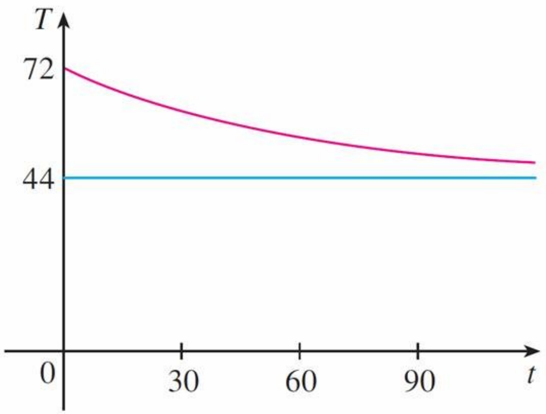


Figure 3

Continuously Compounded Interest

Example 4

If \$1000 is invested at 6% interest, compounded annually, then after 1 year the investment is worth \$1000(1.06) = \$1060, after 2 years it's worth \$[1000(1.06)]1.06 = \$1123.60, and after t years it's worth \$1000$(1.06)^t$.

In general, if an amount A_0 is invested at an interest rate r ($r = 0.06$ in this example), then after t years it's worth $A_0(1 + r)^t$.

Usually, however, interest is compounded more frequently, say, n times a year.

Example 4

cont'd

Then in each compounding period the interest rate is *r*/*n* and there are *nt* compounding periods in *t* years, so the value of the investment is

$$A_0\left(1 + \frac{r}{n}\right)^{nt}$$

For instance, after 3 years at 6% interest a $1000 investment will be worth

$\$1000(1.06)^3 = \$1191.02$ with annual compounding

$\$1000(1.03)^6 = \$1194.05$ with semiannual compounding

Example 4

cont'd

$1000(1.015)^{12}$ = $1195.62 with quarterly compounding

$1000(1.005)^{36}$ = $1196.68 with monthly compounding

$$\$1000\left(1 + \frac{0.06}{365}\right)^{365 \cdot 3} = \$1197.20$$ with daily compounding

You can see that the interest paid increases as the number of compounding periods (n) increases. If we let $n \to \infty$, then we will be compounding the interest **continuously** and the value of the investment will be

$$A(t) = \lim_{n \to \infty} A_0\left(1 + \frac{r}{n}\right)^{nt}$$

Example 4

cont'd

$$= \lim_{n \to \infty} A_0 \left[\left(1 + \frac{r}{n} \right)^{n/r} \right]^{rt}$$

$$= A_0 \left[\lim_{n \to \infty} \left(1 + \frac{r}{n} \right)^{n/r} \right]^{rt}$$

$$= A_0 \left[\lim_{m \to \infty} \left(1 + \frac{1}{m} \right)^{m} \right]^{rt}$$

(where $m = n/r$)

But the limit in this expression is equal to the number *e*.

Example 4

cont'd

So with continuous compounding of interest at interest rate *r*, the amount after *t* years is

$$A(t) = A_0 e^{rt}$$

If we differentiate this equation, we get

$$\frac{dA}{dt} = rA_0 e^{rt} = rA(t)$$

which says that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Example 4

cont'd

Returning to the example of $1000 invested for 3 years at 6% interest, we see that with continuous compounding of interest the value of the investment will be

$$A(3) = \$1000e^{(0.06)3} = \$1197.22$$

Notice how close this is to the amount we calculated for daily compounding, $1197.20. But the amount is easier to compute if we use continuous compounding.