4

Applications of Differentiation

4.1 Maximum and Minimum Values

Maximum and Minimum Values

Some of the most important applications of differential calculus are *optimization problems*, in which we are required to find the optimal (best) way of doing something. These can be done by finding the maximum or minimum values of a function.

Let's first explain exactly what we mean by maximum and minimum values. We see that the highest point on the graph of the function f shown in Figure 1 is the point (3, 5).

In other words, the largest value of f is $f(3) = 5$. Likewise, the smallest value is $f(6) = 2$.

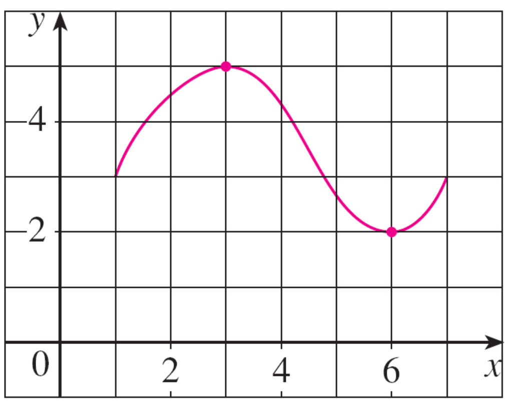


Figure 1

Maximum and Minimum Values

We say that $f(3) = 5$ is the *absolute maximum* of f and $f(6) = 2$ is the *absolute minimum*. In general, we use the following definition.

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

An absolute maximum or minimum is sometimes called a **global** maximum or minimum.

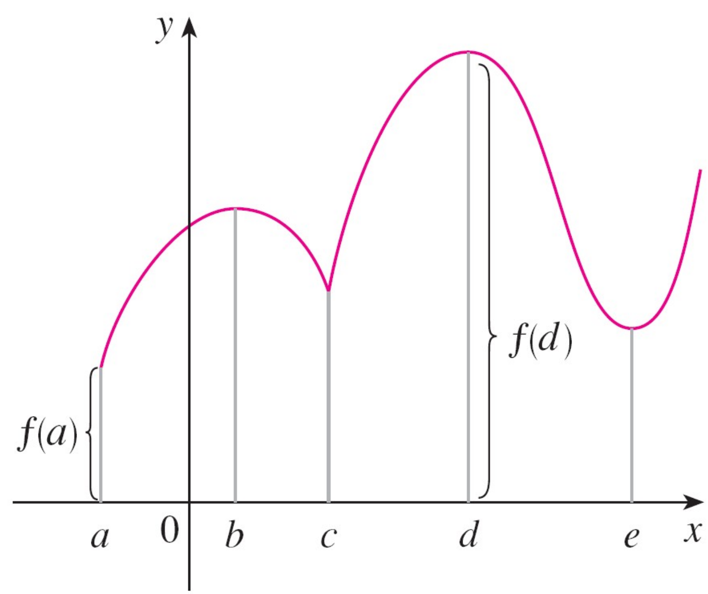
The maximum and minimum values of f are called **extreme values** of f.

Maximum and Minimum Values

Figure 2 shows the graph of a function *f* with absolute maximum at *d* and absolute minimum at *a*.

Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point.

In Figure 2, if we consider only values of *x* near *b* [for instance, if we restrict our attention to the interval (a, c)], then $f(b)$ is the largest of those values of $f(x)$ and is called a *local maximum value* of *f*.



Abs min $f(a)$, abs max $f(d)$
loc min $f(c)$, $f(e)$, loc max $f(b)$, $f(d)$

Figure 2

Maximum and Minimum Values

Likewise, $f(c)$ is called a *local minimum value* of f because $f(c) \le f(x)$ for x near c [in the interval (b, d), for instance].

The function f also has a local minimum at e. In general, we have the following definition.

2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c.

In Definition 2 (and elsewhere), if we say that something is true **near** c, we mean that it is true on some open interval containing c.

Maximum and Minimum Values

For instance, in Figure 3 we see that $f(4) = 5$ is a local minimum because it's the smallest value of f on the interval I.

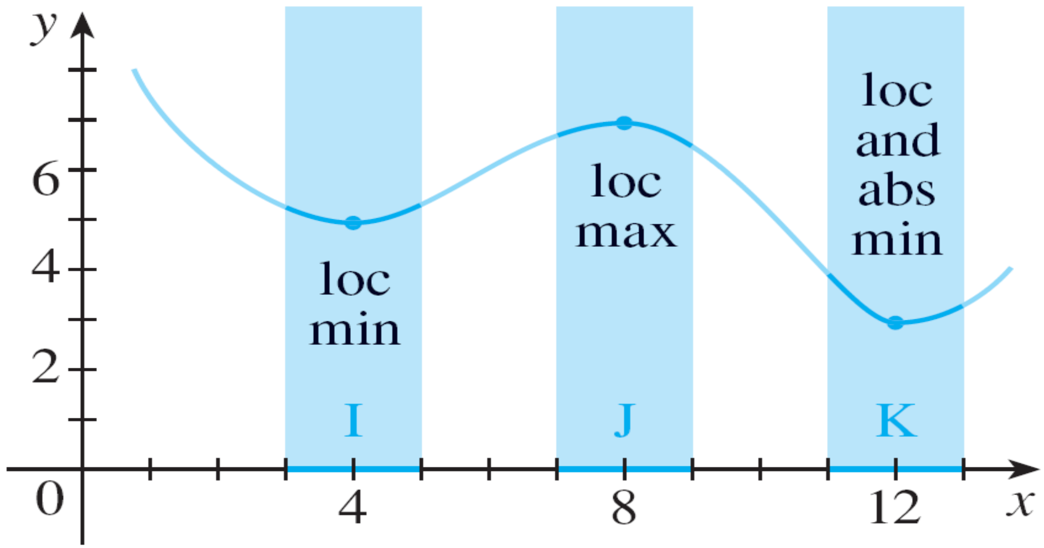


Figure 3

Maximum and Minimum Values

It's not the absolute minimum because $f(x)$ takes smaller values when x is near 12 (in the interval K, for instance).

In fact $f(12) = 3$ is both a local minimum and the absolute minimum.

Similarly, $f(8) = 7$ is a local maximum, but not the absolute maximum because f takes larger values near 1.

Example 1

The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos 2n\pi = 1$ for any integer n and $-1 \leq \cos x \leq 1$ for all x.

Likewise, $\cos(2n + 1)\pi = -1$ is its minimum value, where n is any integer.

Maximum and Minimum Values

The following theorem gives conditions under which a function is guaranteed to possess extreme values.

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Maximum and Minimum Values

The Extreme Value Theorem is illustrated in Figure 7.

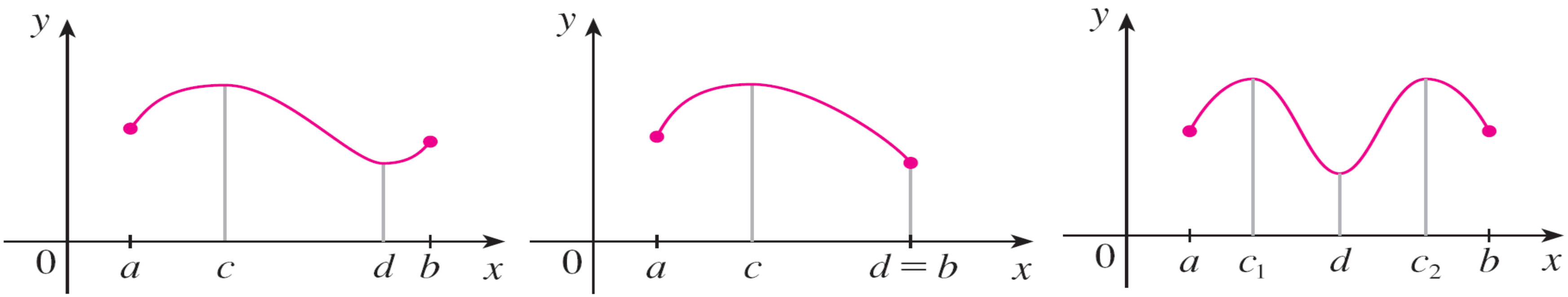
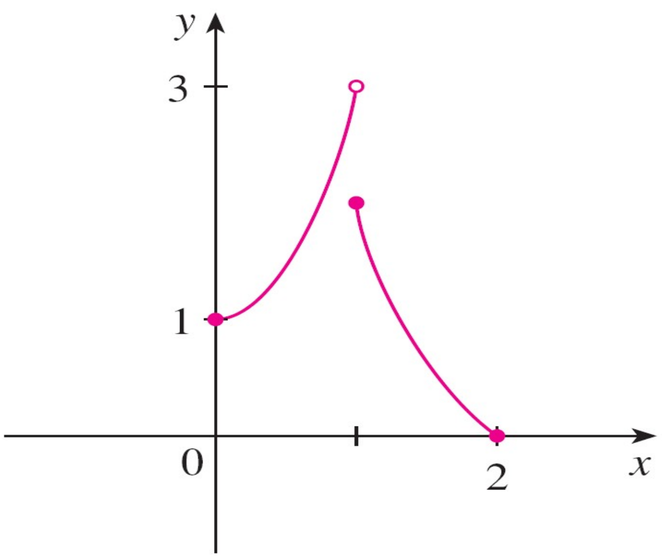


Figure 7

Note that an extreme value can be taken on more than once.

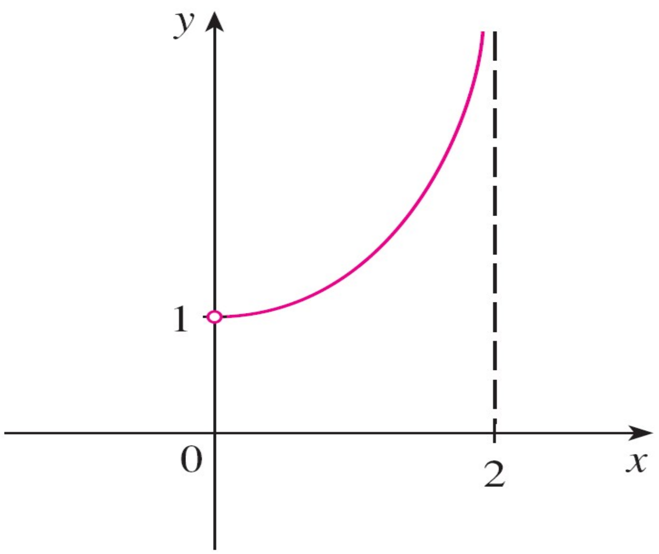
Maximum and Minimum Values

Figures 8 and 9 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.



This function has minimum value
$f(2) = 0$, but no maximum value.

Figure 8



This continuous function g has
no maximum or minimum.

Figure 9

Maximum and Minimum Values

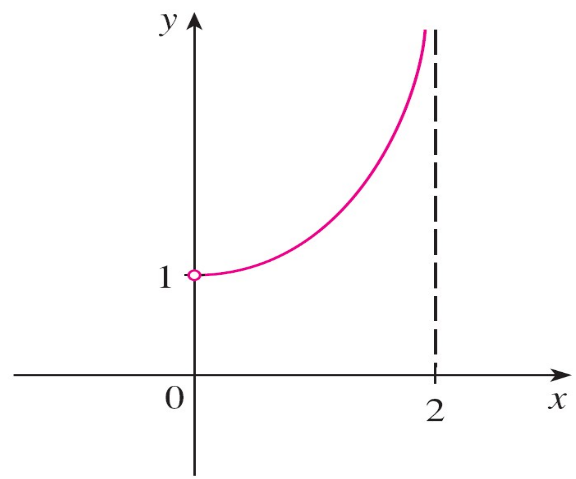
The function *f* whose graph is shown in Figure 8 is defined on the closed interval [0, 2] but has no maximum value. (Notice that the range of *f* is [0, 3). The function takes on values arbitrarily close to 3, but never actually attains the value 3.)

This does not contradict the Extreme Value Theorem because *f* is not continuous.

Maximum and Minimum Values

The function *g* shown in Figure 9 is continuous on the open interval (0, 2) but has neither a maximum nor a minimum value. [The range of *g* is $(1, \infty)$. The function takes on arbitrarily large values.]

This does not contradict the Extreme Value Theorem because the interval (0, 2) is not closed.



This continuous function *g* has
no maximum or minimum.

Figure 9

Maximum and Minimum Values

The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values. We start by looking for local extreme values.

Figure 10 shows the graph of a function *f* with a local maximum at *c* and a local minimum at *d*.

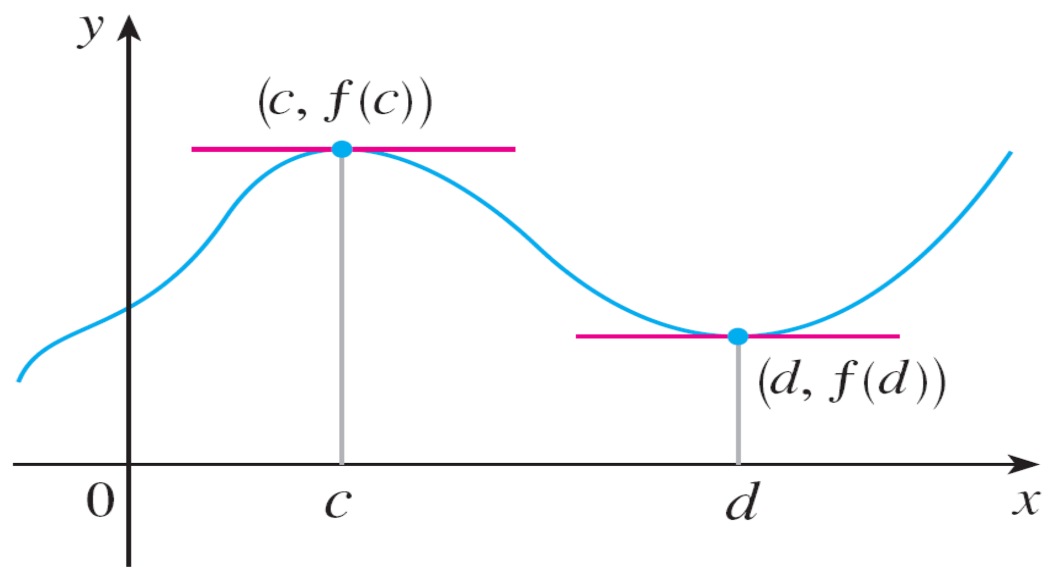


Figure 10

Maximum and Minimum Values

It appears that at the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0.

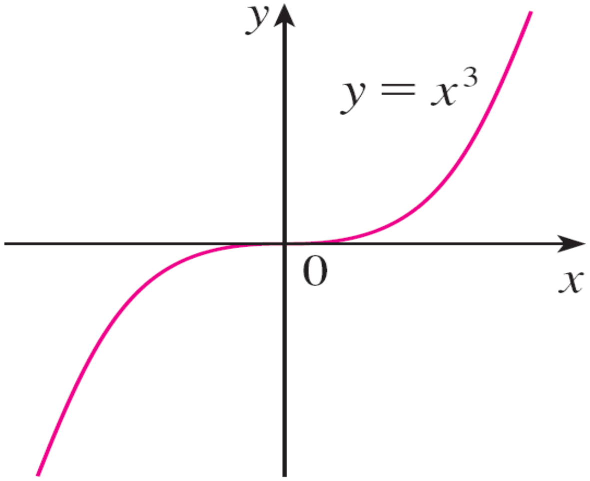
We know that the derivative is the slope of the tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$. The following theorem says that this is always true for differentiable functions.

4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Example 5

If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$.

But f has no maximum or minimum at 0, as you can see from its graph in Figure 11.



If $f(x) = x^3$, then $f'(0) = 0$ but f has no maximum or minimum.

Figure 11

Example 5

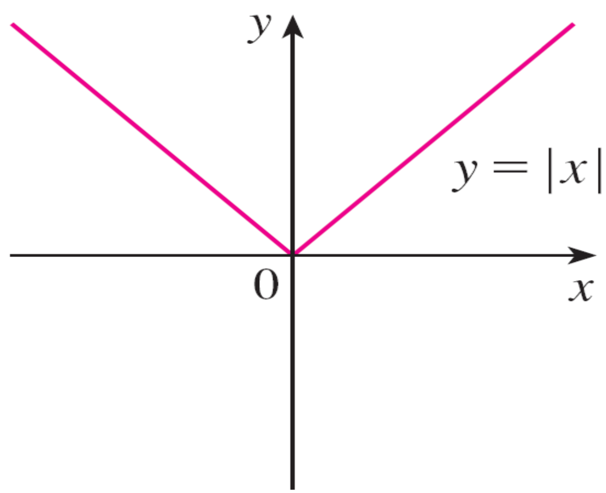
cont'd

The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at (0, 0).

Instead of having a maximum or minimum at (0, 0), the curve crosses its horizontal tangent there.

Example 6

The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value can't be found by setting $f'(x) = 0$ because, $f'(0)$ does not exist. (see Figure 12)



If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

Figure 12

Maximum and Minimum Values

Examples 5 and 6 show that we must be careful when using Fermat's Theorem. Example 5 demonstrates that even when $f'(c) = 0$, f doesn't necessarily have a maximum or minimum at c. (In other words, the converse of Fermat's Theorem is false in general.)

Furthermore, there may be an extreme value even when $f'(c)$ does not exist (as in Example 6).

Maximum and Minimum Values

Fermat's Theorem does suggest that we should at least *start* looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers are given a special name.

6 **Definition** A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

In terms of critical numbers, Fermat's Theorem can be rephrased as follows.

7 If f has a local maximum or minimum at c, then c is a critical number of f.

Maximum and Minimum Values

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local or it occurs at an endpoint of the interval.

Thus the following three-step procedure always works.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.