4

Applications of Differentiation

4.8 Newton's Method

Newton's Method

Suppose that a car dealer offers to sell you a car for \$18,000 or for payments of \$375 per month for five years. You would like to know what monthly interest rate the dealer is, in effect, charging you.

To find the answer, you have to solve the equation

$$48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0 \quad \boxed{1}$$

We can find an *approximate* solution to Equation 1 by plotting the left side of the equation.

Newton’s Method

Using a graphing device, and after experimenting with viewing rectangles, we produce the graph in Figure 1.

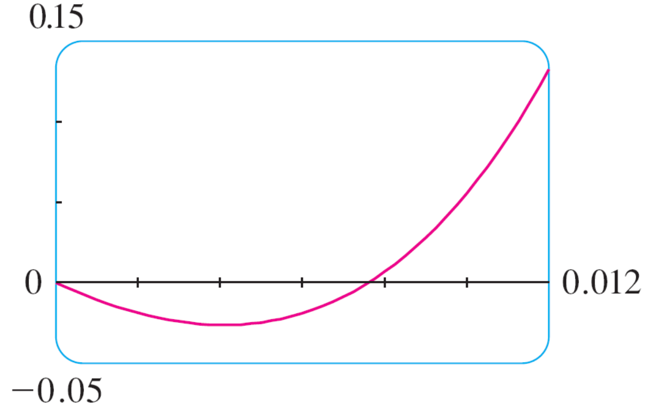

Figure 1

We see that in addition to the solution $x = 0$, which doesn’t interest us, there is a solution between 0.007 and 0.008. Zooming in shows that the root is approximately 0.0076.

If we need more accuracy we could zoom in repeatedly, but that becomes tiresome.

Newton's Method

A faster alternative is to use a numerical rootfinder on a calculator or computer algebra system. If we do so, we find that the root, correct to nine decimal places, is 0.007628603.

How do those numerical rootfinders work? They use a variety of methods, but most of them make some use of **Newton's method**, also called the **Newton-Raphson method**.

We will explain how this method works, partly to show what happens inside a calculator or computer, and partly as an application of the idea of linear approximation.

Newton's Method

The geometry behind Newton's method is shown in Figure 2, where the root that we are trying to find is labeled *r*.

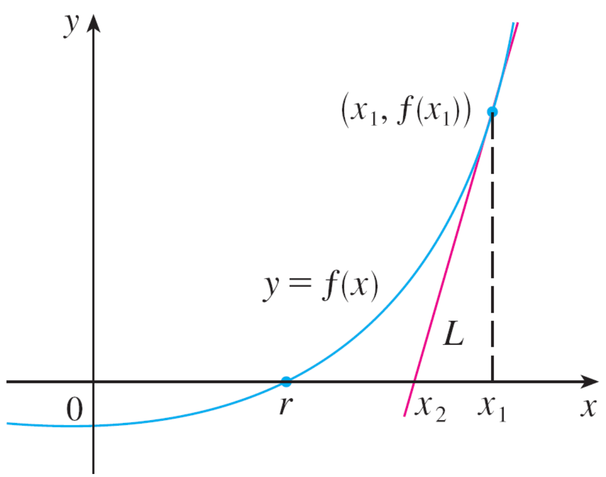

Figure 2

We start with a first approximation x_1, which is obtained by guessing, or from a rough sketch of the graph of *f*, or from a computer-generated graph of *f*.

Newton's Method

Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled x_2.

The idea behind Newton's method is that the tangent line is close to the curve and so its x-intercept, x_2, is close to the x-intercept of the curve (namely, the root r that we are seeking). Because the tangent is a line, we can easily find its x-intercept.

To find a formula for x_2 in terms of x_1 we use the fact that the slope of L is $f'(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Newton's Method

Since the x-intercept of L is x_2, we set $y = 0$ and obtain

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second approximation to r.

Next we repeat this procedure with x_1 replaced by the second approximation x_2, using the tangent line at $(x_2, f(x_2))$.

Newton's Method

This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

If we keep repeating this process, we obtain a sequence of approximations x_1, x_2, x_3, x_4, . . . as shown in Figure 3.

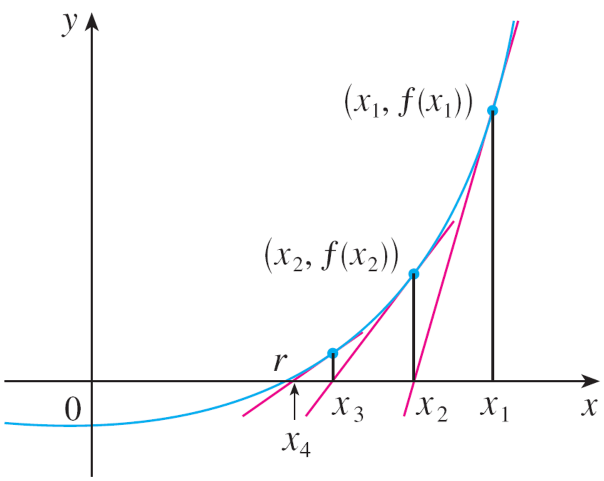

Figure 3

Newton's Method

In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence *converges* to r and we write

$$\lim_{n \to \infty} x_n = r$$

Example 1

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

Solution:

We apply Newton's method with

$$f(x) = x^3 - 2x - 5 \qquad \text{and} \qquad f'(x) = 3x^2 - 2$$

Newton himself used this equation to illustrate his method and he chose $x_1 = 2$ after some experimentation because $f(1) = -6$, $f(2) = -1$, and $f(3) = 16$.

Example 1 – *Solution*

cont'd

Equation 2 becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

With $n = 1$ we have

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2}$$

$$= 2.1$$

Example 1 – *Solution*

cont'd

Then with n = 2 we obtain

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2}$$

$$= 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2}$$

$$\approx 2.0946$$

It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places.