

4

Applications of Differentiation



4.7

Optimization Problems

Optimization Problems

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized.

Let's recall the problem-solving principles.

Optimization Problems

Steps in Solving Optimization Problems

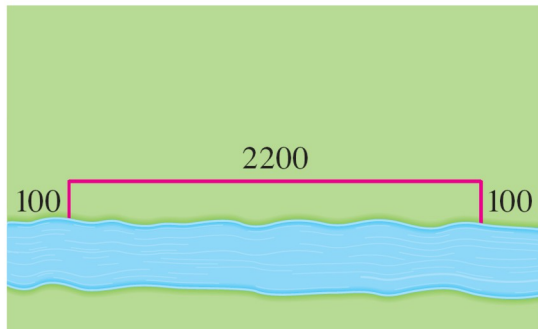
- 1. Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
- 2. Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
- 3. Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols (a, b, c, \dots, x, y) for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.
- 4.** Express Q in terms of some of the other symbols from Step 3.
- 5.** If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q . Thus Q will be expressed as a function of *one* variable x , say, $Q = f(x)$. Write the domain of this function.
- 6.** Use the methods of Sections 3.1 and 3.3 to find the *absolute* maximum or minimum value of f . In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 3.1 can be used.

Example 1

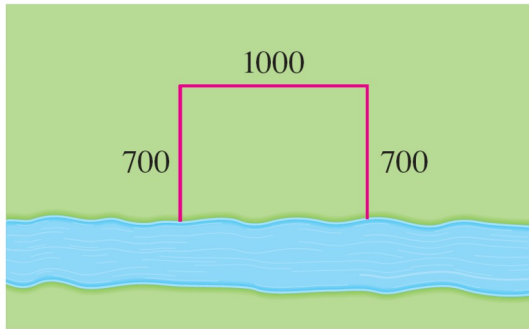
A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Example 1 – Solution

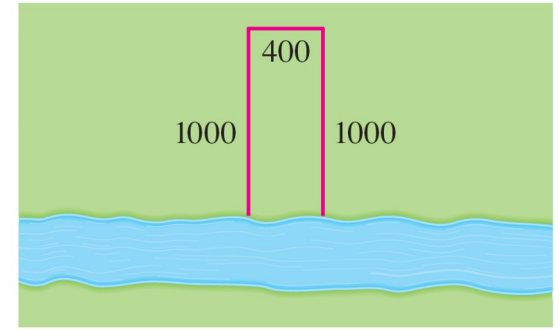
In order to get a feeling for what is happening in this problem, let's experiment with some special cases. Figure 1 (not to scale) shows three possible ways of laying out the 2400 ft of fencing.



$$\text{Area} = 100 \cdot 2200 = 220,000 \text{ ft}^2$$



$$\text{Area} = 700 \cdot 1000 = 700,000 \text{ ft}^2$$



$$\text{Area} = 1000 \cdot 400 = 400,000 \text{ ft}^2$$

Figure 1

Example 1 – *Solution*

cont'd

We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area.

Figure 2 illustrates the general case. We wish to maximize the area A of the rectangle.

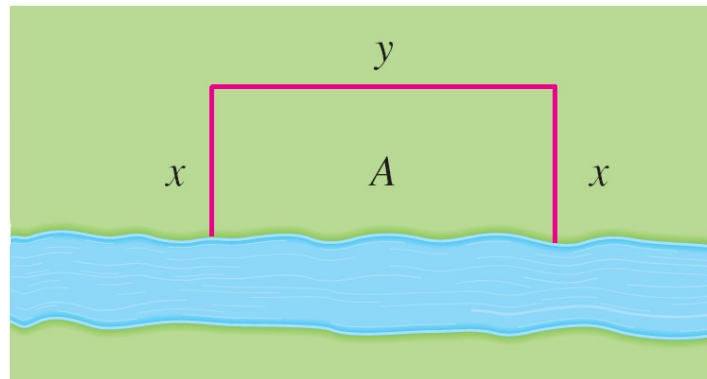


Figure 2

Example 1 – *Solution*

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Let x and y be the depth and width of the rectangle (in feet). Then we express A in terms of x and y :

$$A = xy$$

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x . To do this we use the given information that the total length of the fencing is 2400 ft.

Thus

$$2x + y = 2400$$

From this equation we have $y = 2400 - 2x$, which gives

$$A = x(2400 - 2x) = 2400x - 2x^2$$

Example 1 – *Solution*

cont'd

Note that $x \geq 0$ and $x \leq 1200$ (otherwise $A < 0$). So the function that we wish to maximize is

$$A(x) = 2400x - 2x^2 \quad 0 \leq x \leq 1200$$

The derivative is $A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

which gives $x = 600$.

The maximum value of A must occur either at this critical number or at an endpoint of the interval.

Example 1 – *Solution*

cont'd

Since $A(0) = 0$, $A(600) = 720,000$, and $A(1200) = 0$, the Closed Interval Method gives the maximum value as $A(600) = 720,000$.

[Alternatively, we could have observed that $A''(x) = -4 < 0$ for all x , so A is always concave downward and the local maximum at $x = 600$ must be an absolute maximum.]

Thus the rectangular field should be 600 ft deep and 1200 ft wide.

Optimization Problems

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

- (a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f .
- (b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f .



Applications to Business and Economics

Applications to Business and Economics

We know that if $C(x)$, the **cost function**, is the cost of producing x units of a certain product, then the **marginal cost** is the rate of change of C with respect to x .

In other words, the marginal cost function is the derivative, $C'(x)$, of the cost function.

Now let's consider marketing. Let $p(x)$ be the price per unit that the company can charge if it sells x units.

Then p is called the **demand function** (or **price function**) and we would expect it to be a decreasing function of x .

Applications to Business and Economics

If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = xp(x)$$

and R is called the **revenue function**.

The derivative R' of the revenue function is called the **marginal revenue function** and is the rate of change of revenue with respect to the number of units sold.

Applications to Business and Economics

If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**.

The **marginal profit function** is P' , the derivative of the profit function.

Example 6

A store has been selling 200 Blu-ray disc players a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

Solution:

If x is the number of Blu-ray players sold per week, then the weekly increase in sales is $x - 200$.

For each increase of 20 units sold, the price is decreased by \$10.

Example 6 – *Solution*

cont'd

So for each additional unit sold, the decrease in price will be $\frac{1}{20} \times 10$ and the demand function is

$$p(x) = 350 - \frac{10}{20}(x - 200) = 450 - \frac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \frac{1}{2}x^2$$

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$.

This value of x gives an absolute maximum by the First Derivative Test (or simply by observing that the graph of R is a parabola that opens downward).

Example 6 – *Solution*

cont'd

The corresponding price is

$$p(450) = 450 - \frac{1}{2}(450) = 225$$

and the rebate is $350 - 225 = 125$.

Therefore, to maximize revenue, the store should offer a rebate of \$125.