4 Applications of Differentiation

4.9 Antiderivatives

Antiderivatives

A physicist who knows the velocity of a particle might wish to know its position at a given time.

An engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period.

A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the size of the population will be at some future time.

Antiderivatives

In each case, the problem is to find a function F whose derivative is a known function f. If such a function F exists, it is called an *antiderivative* of f.

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Antiderivatives

For instance, let $f(x) = x^2$. It isn't difficult to discover an antiderivative of f if we keep the Power Rule in mind. In fact, if $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$.

But the function $G(x) = \frac{1}{3}x^3 + 100$ also satisfies $G'(x) = x^2$. Therefore both F and G are antiderivatives of f.

Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f.

Antiderivatives

The following theorem says that f has no other antiderivative

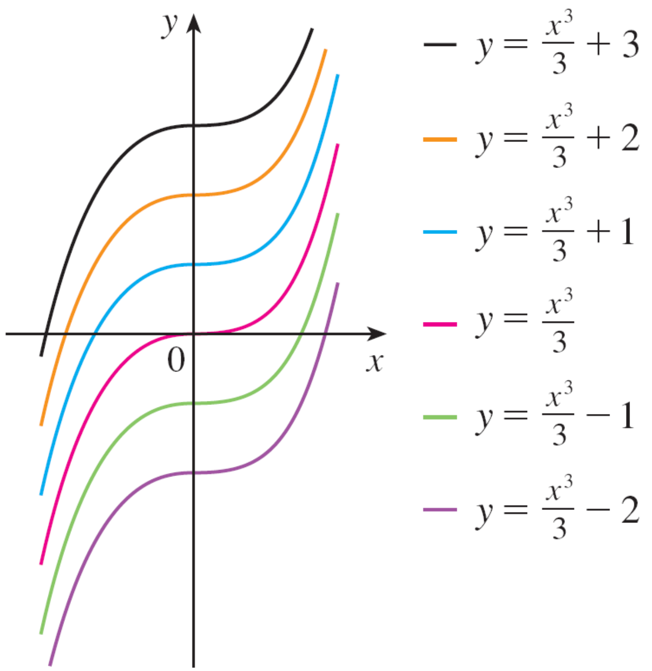
> **1 Theorem** If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is
>
> $$F(x) + C$$
>
> where C is an arbitrary constant.

Going back to the function $f(x) = x^2$, we see that the general antiderivative of f is $x^3/3 + C$.

Antiderivatives

By assigning specific values to the constant C, we obtain a family of functions whose graphs are vertical translates of one another (see Figure 1).

This makes sense because each curve must have the same slope at any given value of x.



Members of the family of antiderivatives of $f(x) = x^2$

Figure 1

Example 1

Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$ **(b)** $f(x) = 1/x$ **(c)** $f(x) = x^n$, $n \neq -1$

Solution:

(a) If $F(x) = -\cos x$, then $F'(x) = \sin x$, so an antiderivative of $\sin x$ is $-\cos x$.

By Theorem 1, the most general antiderivative is $G(x) = -\cos x + C$.

Example 1 – *Solution*

cont'd

(b) Recall

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

So on the interval $(0, \infty)$ the general antiderivative of 1/x is ln x + C. We also learned that

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}$$

for all $x \neq 0$. Theorem 1 then tells us that the general antiderivative of $f(x) = 1/x$ is ln $|x| + C$ on any interval that doesn't contain 0. In particular, this is true on each of the intervals $(-\infty, 0)$ and $(0, \infty)$.

Example 1 – *Solution*

cont'd

So the general antiderivative of *f* is

$$F(x) = \begin{cases} \ln x + C_1 & \text{if } x > 0 \\ \ln(-x) + C_2 & \text{if } x < 0 \end{cases}$$

(c) We use the Power Rule to discover an antiderivative of x^n. In fact, if $n \neq -1$, then

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

Thus the general antiderivative of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

Example 1 – *Solution*

cont'd

This is valid for $n \geq 0$ since then $f(x) = x^n$ is defined on an interval. If n is negative (but $n \neq -1$), it is valid on any interval that doesn't contain 0.

Antiderivatives

As in Example 1, every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula. In Table 2 we list some particular antiderivatives.

2 Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sec^2 x$	$\tan x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec x \tan x$	$\sec x$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\dfrac{1}{x}$	$\ln \lvert x \rvert$	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
e^x	e^x	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$
$\sin x$	$-\cos x$		

To obtain the most general anti derivative from the particular ones in Table 2, we have to add a constant (or constants), as in Example 1.

Antiderivatives

Each formula in the table is true because the derivative of the function in the right column appears in the left column.

In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function.

The second formula says that the antiderivative of a sum is the sum of the antiderivatives. (We use the notation $F' = f$, $G' = g$.)

Antiderivatives

An equation that involves the derivatives of a function is called a **differential equation**.

The general solution of a differential equation involves an arbitrary constant (or constants).

However, there may be some extra conditions given that will determine the constants and therefore uniquely specify the solution.

Rectilinear Motion

Rectilinear Motion

Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line. Recall that if the object has position function $s = f(t)$, then the velocity function is $v(t) = s'(t)$.

This means that the position function is an antiderivative of the velocity function.

Likewise, the acceleration function is $a(t) = v'(t)$, so the velocity function is an antiderivative of the acceleration.

If the acceleration and the initial values $s(0)$ and $v(0)$ are known, then the position function can be found by antidifferentiating twice.

Example 6

A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Solution:

Since $v'(t) = a(t) = 6t + 4$, antidifferentiation gives

$$v(t) = 6\frac{t^2}{2} + 4t + C$$

$$= 3t^2 + 4t + C$$

Example 6 – *Solution*

cont'd

Note that $v(0) = C$. But we are given that $v(0) = -6$, so $C = -6$ and

$$v(t) = 3t^2 + 4t - 6$$

Since $v(t) = s'(t)$, s is the antiderivative of v:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

This gives $s(0) = D$. We are given that $s(0) = 9$, so $D = 9$ and the required position function is

$$s(t) = t^3 + 2t^2 - 6t + 9$$