5

Integrals

5.2 The Definite Integral

The Definite Integral

We have seen that a limit of the form

$$\boxed{1} \quad \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n \to \infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

arises when we compute an area.

We also have seen that it arises when we try to find the distance traveled by an object.

It turns out that this same type of limit occurs in a wide variety of situations even when *f* is not necessarily a positive function.

The Definite Integral

2 Definition of a Definite Integral If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 (= a), x_1, x_2, \ldots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Note 1: The symbol $\int$ was introduced by Leibniz and is called an **integral sign**.

It is an elongated *S* and was chosen because an integral is a limit of sums.

The Definite Integral

In the notation $\int_a^b f(x)\,dx$, *f*(*x*) is called the **integrand** and *a* and *b* are called the **limits of integration**; *a* is the **lower limit** and *b* is the **upper limit**.

For now, the symbol *dx* has no meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol.

The *dx* simply indicates that the independent variable is *x*. The procedure of calculating an integral is called **integration**.

The Definite Integral

Note 2: The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on *x*. In fact, we could use any letter in place of *x* without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

Note 3: The sum

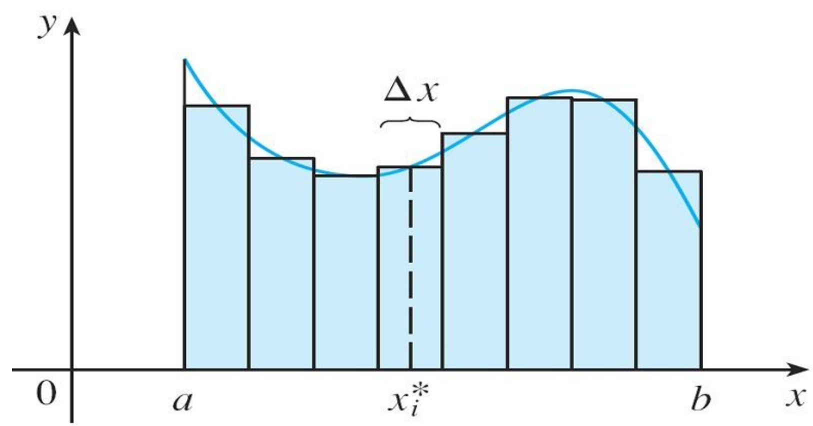
$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866).

The Definite Integral

So Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.

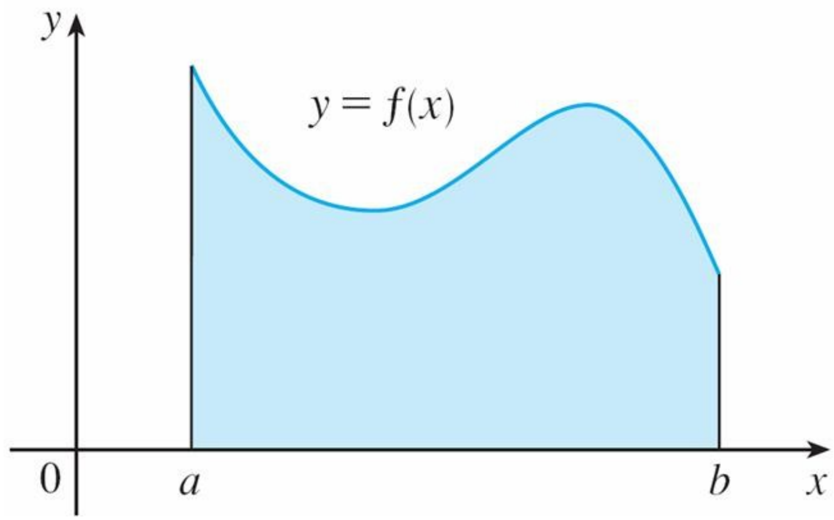
We know that if *f* happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1).



If $f(x) \geq 0$, the Riemann sum $\Sigma f(x_i^*)$
Δx
is the sum of areas of rectangles.
Figure 1

The Definite Integral

We see that the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b. (See Figure 2.)

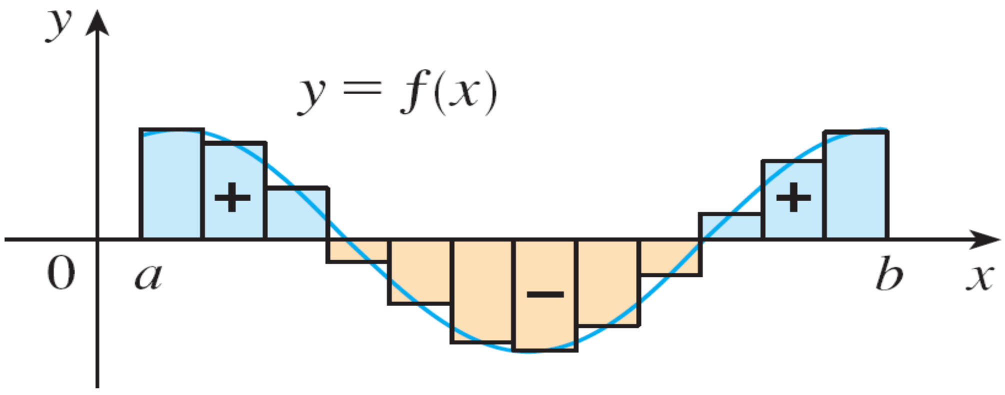


If $f(x) \geq 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

Figure 2

The Definite Integral

If *f* takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the *x*-axis and the *negatives* of the areas of the rectangles that lie below the *x*-axis (the areas of the blue rectangles *minus* the areas of the gold rectangles).



$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area.

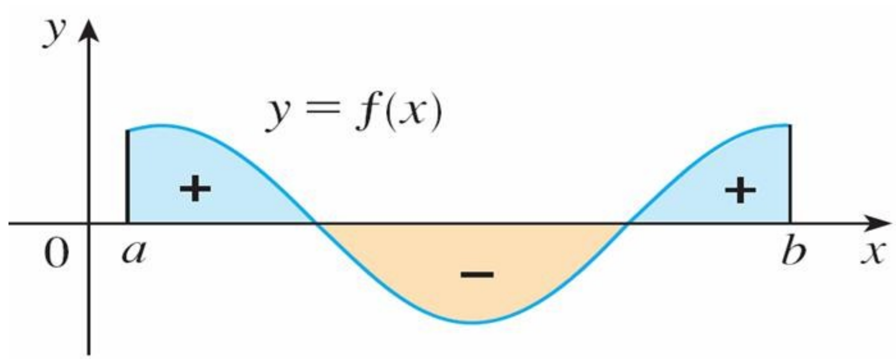
Figure 3

The Definite Integral

When we take the limit of such Riemann sums, we get the situation illustrated in Figure 4. A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the *x*-axis and below the graph of *f*, and A_2 is the area of the region below the *x*-axis and above the graph of *f*.



$\int_a^b f(x)\,dx$ is the net area.

Figure 4

The Definite Integral

Note 4: Although we have defined $\int_a^b f(x)\,dx$ by dividing [*a*, *b*] into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width.

If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, max Δx_i, approaches 0. So in this case the definition of a definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

The Definite Integral

Note 5: We have defined the definite integral for an integrable function, but not all functions are integrable. The following theorem shows that the most commonly occurring functions are in fact integrable. It is proved in more advanced courses.

3 Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If f is integrable on $[a, b]$, then the limit in Definition 2 exists and gives the same value no matter how we choose the sample points x_i^*.

The Definite Integral

To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$

Example 1

Express

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x$$

as an integral on the interval $[0, \pi]$.

Solution:

Comparing the given limit with the limit in Theorem 4, we see that they will be identical if we choose $f(x) = x^3 + x \sin x$. We are given that $a = 0$ and $b = \pi$.

Therefore, by Theorem 4, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x = \int_0^{\pi} (x^3 + x \sin x)\,dx$$

The Definite Integral

When Leibniz chose the notation for an integral, he chose the ingredients as reminders of the limiting process.

In general, when we write

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x = \int_a^b f(x)\, dx$$

we replace lim Σ by $\int$, x_i^* by x, and Δx by dx.

Evaluating Integrals

Evaluating Integrals

When we use a limit to evaluate a definite integral, we need to know how to work with sums. The following three equations give formulas for sums of powers of positive integers.

5 $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

6 $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

7 $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Evaluating Integrals

The remaining formulas are simple rules for working with sigma notation:

8 $$\sum_{i=1}^{n} c = nc$$

9 $$\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$$

10 $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

11 $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Example 2 – *Evaluating an integral as a limit of Riemann sums*

(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the

sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b)(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx.$

(c)Solution:

(d) With $n = 6$ the interval width is

$$\Delta x = \frac{b - a}{n} = \frac{3 - 0}{6} = \frac{1}{2}$$

(e) and the right endpoints are $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$,

Example 2 – *Solution*

cont'd

So the Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$$

$$= f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x + f(2.0)\,\Delta x + f(2.5)\,\Delta x + f(3.0)\,\Delta x$$

$$= \frac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9)$$

$$= -3.9375$$

Example 2 – *Solution*

cont'd

Notice that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas of the blue rectangles (above the x-axis) minus the sum of the areas of the gold rectangles (below the x-axis) in Figure 5.

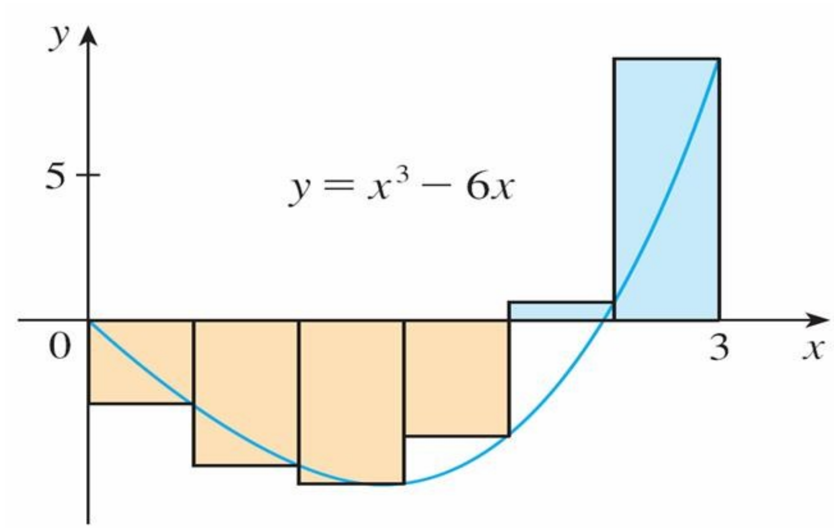


Figure 5

Example 2 – *Solution*

cont'd

(b) With *n* subintervals we have

$$\Delta x = \frac{b - a}{n} = \frac{3}{n}$$

Thus $x_0 = 0$, $x_1 = 3/n$, $x_2 = 6/n$, $x_3 = 9/n$, and, in general, $x_i = 3i/n$.

Since we are using right endpoints, we can use Theorem 4:

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

Example 2 – *Solution*

cont'd

$$= \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n} \right)^3 - 6\left(\frac{3i}{n} \right) \right]$$ (Equation 9 with $c = 3/n$)

$$= \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i \right]$$

$$= \lim_{n \to \infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i \right]$$ (Equations 11 and 9)

$$= \lim_{n \to \infty} \left\{ \frac{81}{n^4} \left[\frac{n(n+1)}{2} \right]^2 - \frac{54}{n^2} \frac{n(n+1)}{2} \right\}$$ (Equations 7 and 5)

Example 2 – *Solution*

cont'd

$$= \lim_{n \to \infty} \left[\frac{81}{4} \left(1 + \frac{1}{n} \right)^2 - 27 \left(1 + \frac{1}{n} \right) \right]$$

$$= \frac{81}{4} - 27$$

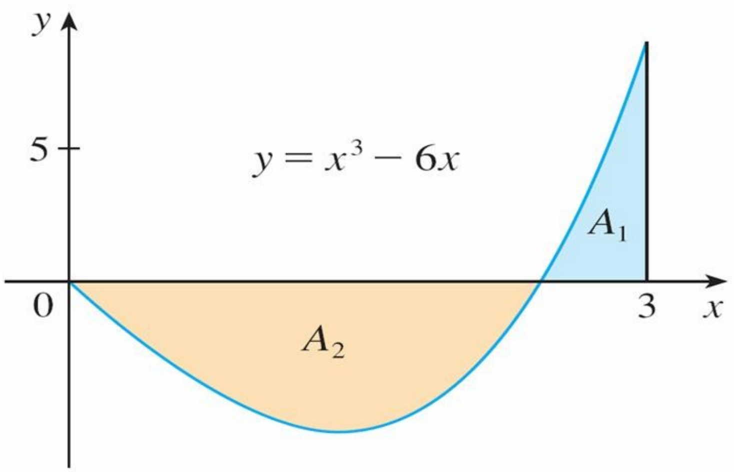
$$= -\frac{27}{4}$$

$$= -6.75$$

Example 2 – *Solution*

cont'd

This integral can't be interpreted as an area because f takes on both positive and negative values. But it can be interpreted as the difference of areas $A_1 - A_2$, where A_1 and A_2 are shown in Figure 6.



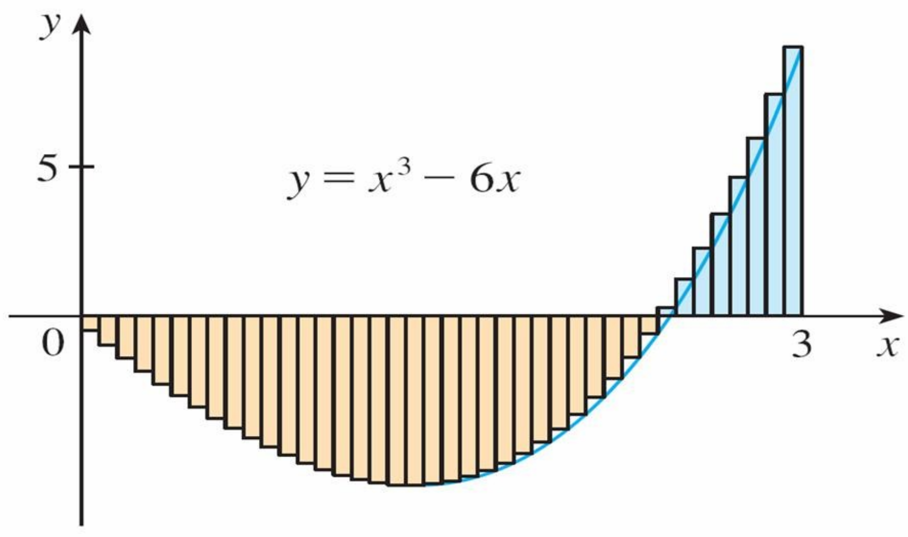
$$\int_0^3 (x^3 - 6x)\,dx = A_1 - A_2 = -6.75$$

Figure 6

Example 2 – *Solution*

cont'd

Figure 7 illustrates the calculation by showing the positive and negative terms in the right Riemann sum R_n for $n = 40$.



$R_{40} \approx -6.3998$

Figure 7

Example 2 – *Solution*

cont'd

The values in the table show the Riemann sums approaching the exact value of the integral, –6.75, as $n \to \infty$.

n	R_n
40	−6.3998
100	−6.6130
500	−6.7229
1000	−6.7365
5000	−6.7473

The Midpoint Rule

The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the *i*th subinterval because it is convenient for computing the limit.

But if the purpose is to find an *approximation* to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\overline{x}_i$.

The Midpoint Rule

Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\,\Delta x = \Delta x\,[f(\overline{x}_1) + \cdots + f(\overline{x}_n)]$$

where $$\Delta x = \frac{b-a}{n}$$

and $$\overline{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

Example 5

Use the Midpoint Rule with n = 5 to approximate $\int_1^2 \frac{1}{x}\,dx.$

Solution:

The endpoints of the five subintervals are 1, 1.2, 1.4, 1.6, 1.8, and 2.0, so the midpoints are 1.1, 1.3, 1.5, 1.7, and 1.9.

The width of the subintervals is Δx = (2 – 1)/5 = $\frac{1}{5}$, so the Midpoint Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\,[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

Example 5 – *Solution*

cont'd

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx 0.691908$$

Since $f(x) = 1/x > 0$ for $1 \leq x \leq 2$, the integral represents an area, and the approximation given by the Midpoint Rule is the sum of the areas of the rectangles shown in Figure 11.

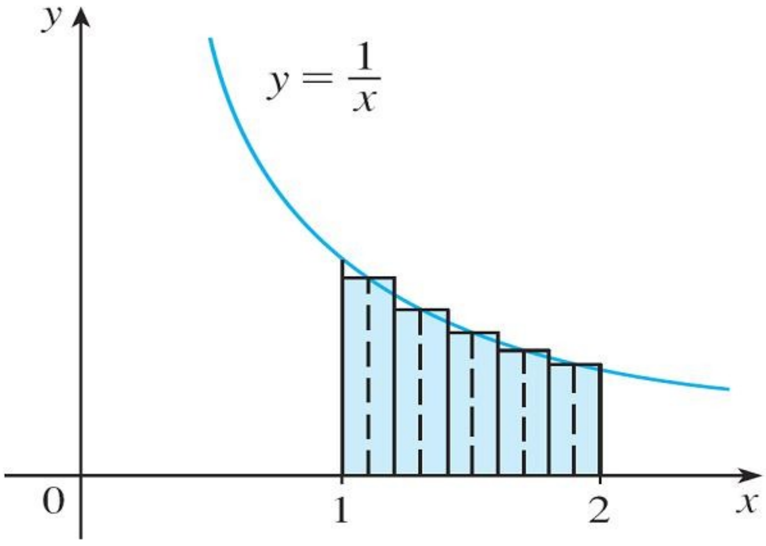


Figure 11

Properties of the Definite Integral

Properties of the Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$.

But the definition as a limit of Riemann sums makes sense even if $a > b$.

Notice that if we reverse a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Properties of the Definite Integral

If $a = b$, then $\Delta x = 0$ and so

$$\int_a^a f(x)\,dx = 0$$

We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that *f* and *g* are continuous functions.

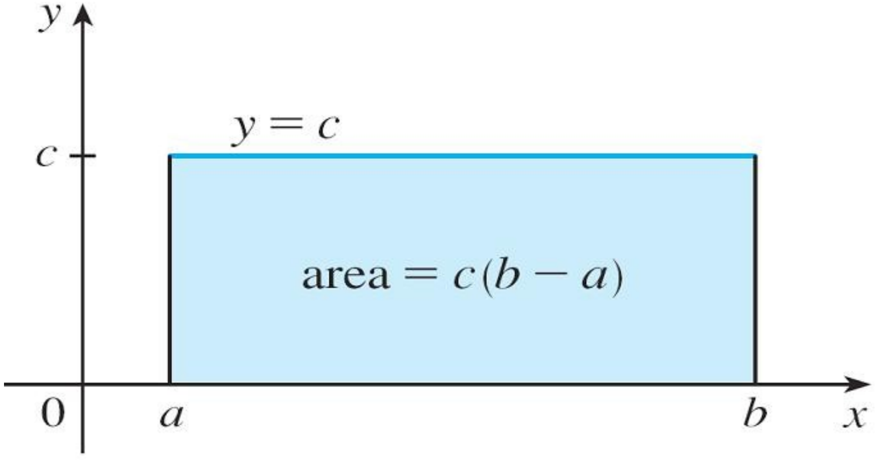
Properties of the Integral

1. $\int_a^b c\,dx = c(b - a)$, where c is any constant
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Properties of the Definite Integral

Property 1 says that the integral of a constant function $f(x) = c$ is the constant times the length of the interval.

If $c > 0$ and $a < b$, this is to be expected because $c(b - a)$ is the area of the shaded rectangle in Figure 13.



$$\int_a^b c\,dx = c(b-a)$$

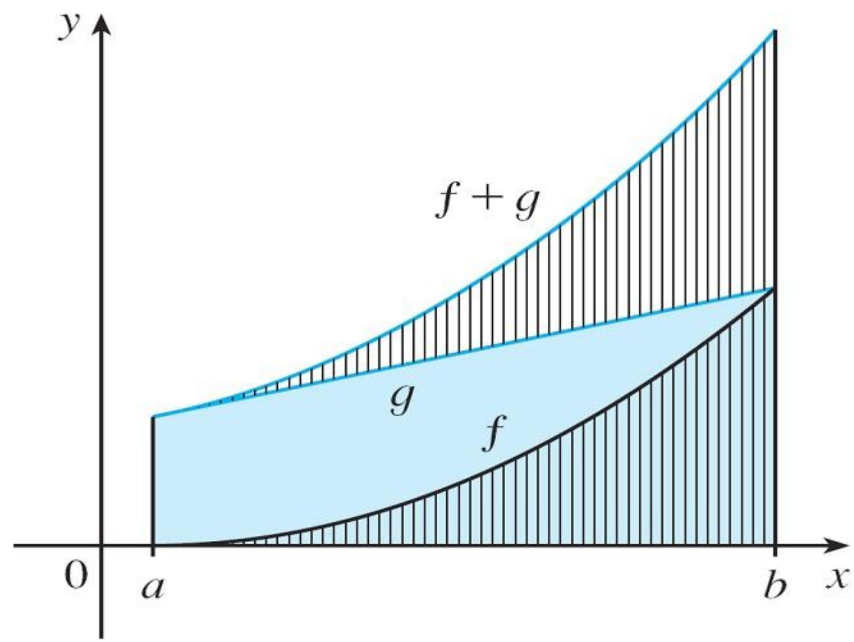
Figure 13

Properties of the Definite Integral

Property 2 says that the integral of a sum is the sum of the integrals.

For positive functions it says that the area under *f* + *g* is the area under *f* plus the area under *g*.

Figure 14 helps us understand why this is true: In view of how graphical addition works, the corresponding vertical line segments have equal height.



$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Figure 14

Properties of the Definite Integral

In general, Property 2 follows from Theorem 4 and the fact that the limit of a sum is the sum of the limits:

$$\int_a^b [f(x) + g(x)]\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\,\Delta x$$

$$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} f(x_i)\,\Delta x + \sum_{i=1}^{n} g(x_i)\,\Delta x \right]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x + \lim_{n \to \infty} \sum_{i=1}^{n} g(x_i)\,\Delta x$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Properties of the Definite Integral

Property 3 can be proved in a similar manner and says that the integral of a constant times a function is the constant times the integral of the function.

In other words, a constant (but *only* a constant) can be taken in front of an integral sign.

Property 4 is proved by writing $f - g = f + (-g)$ and using Properties 2 and 3 with $c = -1$.

Example 6

Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

Solution:

Using Properties 2 and 3 of integrals, we have

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx$$

$$= \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

Example 6 – *Solution*

cont'd

We know from Property 1 that

$$\int_0^1 4\,dx = 4(1 - 0)$$

and we have found that $\int_0^1 x^2\,dx = \frac{1}{3}$.

So

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

$$= 4 + 3 \cdot \tfrac{1}{3}$$

$$= 5$$

Properties of the Definite Integral

The next property tells us how to combine integrals of the same function over adjacent intervals:

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Properties of the Definite Integral

This is not easy to prove in general, but for the case where $f(x) \geq 0$ and $a < c < b$ Property 5 can be seen from the geometric interpretation in Figure 15: The area under $y = f(x)$ from a to c plus the area from c to b is equal to the total area from a to b.

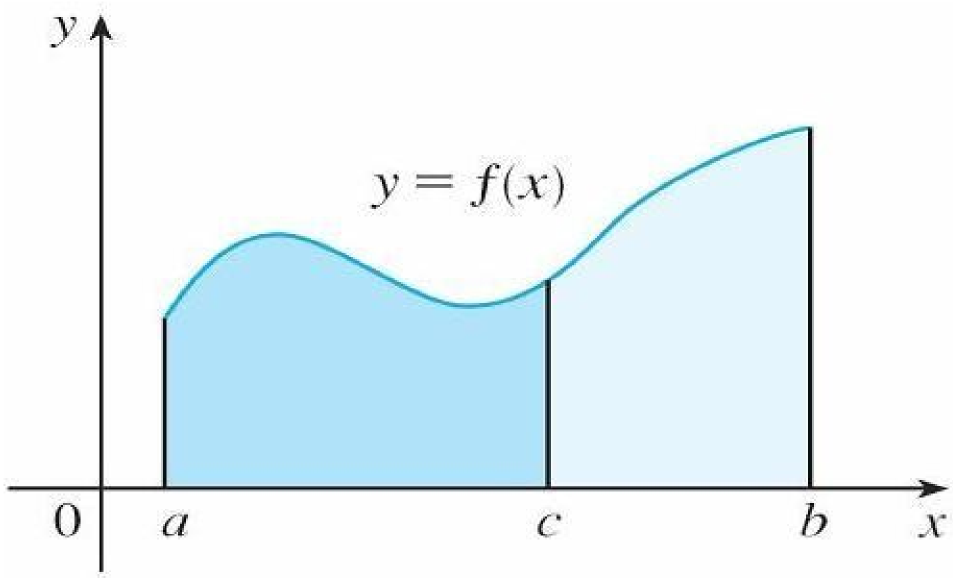


Figure 15

Properties of the Definite Integral

Properties 1–5 are true whether $a < b$, $a = b$, or $a > b$. The following properties, in which we compare sizes of functions and sizes of integrals, are true only if $a \leq b$.

Comparison Properties of the Integral

6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$$

Properties of the Definite Integral

If $f(x) \geq 0$, then $\int_a^b f(x)\,dx$ represents the area under the graph of f, so the geometric interpretation of Property 6 is simply that areas are positive. (It also follows directly from the definition because all the quantities involved are positive.)

Property 7 says that a bigger function has a bigger integral.

It follows from Properties 6 and 4 because $f - g \geq 0$.

Properties of the Definite Integral

Property 8 is illustrated by Figure 16 for the case where $f(x) \geq 0$.

If *f* is continuous we could take *m* and *M* to be the absolute minimum and maximum values of *f* on the interval [a, b].

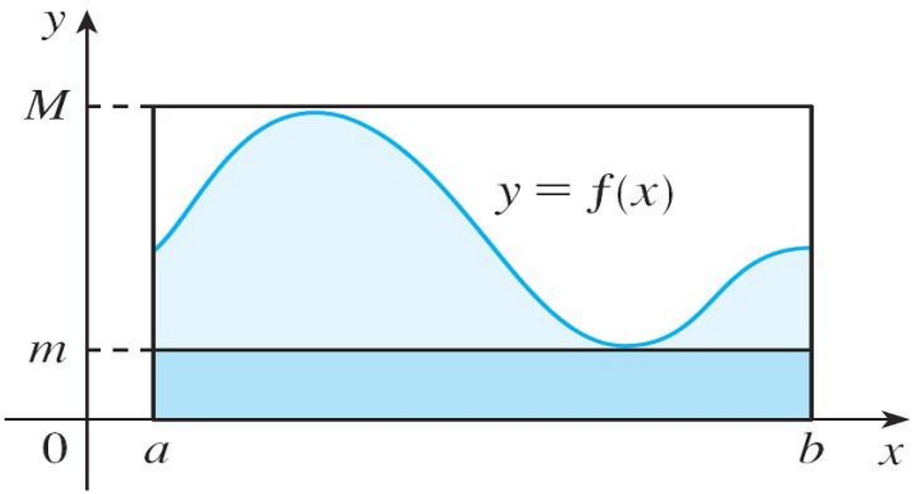


Figure 16

Properties of the Definite Integral

In this case Property 8 says that the area under the graph of *f* is greater than the area of the rectangle with height *m* and less than the area of the rectangle with height *M*.

Property 8 is useful when all we want is a rough estimate of the size of an integral without going to the bother of using the Midpoint Rule.