5 Integrals

5.5 The Substitution Rule

The Substitution Rule

Because of the Fundamental Theorem, it's important to be able to find antiderivatives.

But our antidifferentiation formulas don't tell us how to evaluate integrals such as

$$\boxed{1} \qquad \int 2x\sqrt{1 + x^2}\,dx$$

To find this integral we use the problem-solving strategy of *introducing something extra.* Here the "something extra" is a new variable; we change from the variable *x* to a new variable *u*.

The Substitution Rule

Suppose that we let u be the quantity under the root sign in $\boxed{1}$

, $u = 1 + x^2$. Then the differential of u is $du = 2xdx$.

Notice that if the dx in the notation for an integral were to be interpreted as $\boxed{1}$ a differential, then the differential $2xdx$ would occur in and so, formally, without justifying our calculation, we could write

$$\boxed{2} \quad \int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx = \int \sqrt{u}\,du$$

$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(x^2+1)^{3/2} + C$$

The Substitution Rule

But now we can check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx}\left[\tfrac{2}{3}(x^2 + 1)^{3/2} + C\right] = \tfrac{2}{3} \cdot \tfrac{3}{2}(x^2 + 1)^{1/2} \cdot 2x = 2x\sqrt{x^2 + 1}$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))\, g'(x)\, dx$.

The Substitution Rule

Observe that if $F' = f$, then

3 $$\int F'(g(x))\, g'(x)\, dx = F(g(x)) + C$$

because, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

If we make the "change of variable" or "substitution" $u = g(x)$, then from Equation 3 we have

$$\int F'(g(x))g'(x)\, dx = F(g(x)) + C = F(u) + C = \int F'(u)\, du$$

or, writing $F' = f$, we get

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du$$

The Substitution Rule

Thus we have proved the following rule.

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Notice that the Substitution Rule for integration was proved using the Chain Rule for differentiation.

Notice also that if $u = g(x)$, then $du = g'(x)\,dx$, so a way to remember the Substitution Rule is to think of dx and du in

4 as differentials.

The Substitution Rule

Thus the Substitution Rule says: **It is permissible to operate with *dx* and *du* after integral signs as if they were differentials.**

Example 1

Find $\int x^3 \cos(x^4 + 2)\, dx$.

Solution:

We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\, dx$, which, apart from the constant factor 4, occurs in the integral.

Thus, using $x^3\, dx = \frac{1}{4} du$ and the Substitution Rule, we have

$$\int x^3 \cos(x^4 + 2)\, dx = \int \cos u \cdot \tfrac{1}{4} du$$

$$= \tfrac{1}{4} \int \cos u\, du$$

Example 1 – *Solution*

cont'd

$$= \frac{1}{4} \sin u + C$$

$$= \frac{1}{4} \sin(x^4 + 2) + C$$

Notice that at the final stage we had to return to the original variable x.

Definite Integrals

Definite Integrals

When evaluating a *definite* integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Fundamental Theorem.

For example,

$$\int_0^4 \sqrt{2x+1}\,dx = \int \sqrt{2x+1}\,dx\Big]_0^4 = \tfrac{1}{3}(2x+1)^{3/2}\Big]_0^4$$

$$= \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2} = \tfrac{1}{3}(27-1) = \tfrac{26}{3}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

Definite Integrals

6 **The Substitution Rule for Definite Integrals** If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Example 7

Evaluate $\int_0^4 \sqrt{2x + 1}\, dx$ using $\boxed{6}$.

Solution:

Let $u = 2x + 1$. Then $du = 2\, dx$, so $dx = \frac{1}{2} du$.

To find the new limits of integration we note that

when $x = 0$, $u = 2(0) + 1 = 1$

and

when $x = 4$, $u = 2(4) + 1 = 9$

Example 7 – *Solution*

cont'd

Therefore

$$\int_0^4 \sqrt{2x + 1}\, dx = \int_1^9 \tfrac{1}{2} \sqrt{u}\, du$$

$$= \tfrac{1}{2} \cdot \tfrac{2}{3} u^{3/2} \Big]_1^9$$

$$= \tfrac{1}{3}(9^{3/2} - 1^{3/2})$$

$$= \frac{26}{3}$$

Observe that when using [6] we do *not* return to the variable *x* after integrating. We simply evaluate the expression in *u* between the appropriate values of *u*.

Symmetry

Symmetry

The following theorem uses the Substitution Rule for Definite Integrals [6] to simplify the calculation of integrals of functions that possess symmetry properties.

[7] Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Symmetry

Theorem 7 is illustrated by Figure 3.

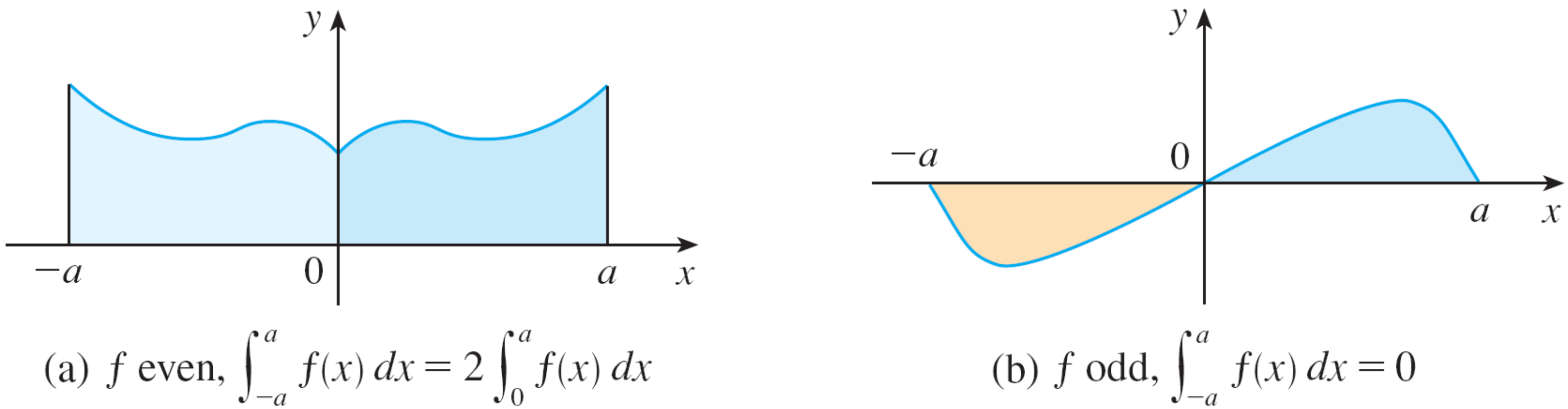

Figure 3

For the case where *f* is positive and even, part (a) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry.

Symmetry

We know that an integral $\int_a^b f(x)\,dx$ can be expressed as the area above the *x*-axis and below $y = f(x)$ minus the area below the axis and above the curve.

Thus part (b) says the integral is 0 because the areas cancel.

Example 10

Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_{0}^{2} (x^6 + 1)\,dx$$

$$= 2\left[\tfrac{1}{7}x^7 + x\right]_0^2$$

$$= 2\left(\tfrac{128}{7} + 2\right)$$

$$= \frac{284}{7}$$

Example 11

Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\, dx = 0$$