5 Integrals

5.4 Indefinite Integrals and the Net Change Theorem

Indefinite Integrals and the Net Change Theorem

In this section we introduce a notation for antiderivatives, review the formulas for antiderivatives, and use them to evaluate definite integrals.

We also reformulate FTC2 in a way that makes it easier to apply to science and engineering problems.

Indefinite Integrals

Indefinite Integrals

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals. Part 1 says that if *f* is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of *f*. Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating $F(b) - F(a)$, where *F* is an antiderivative of *f*.

We need a convenient notation for antiderivatives that makes them easy to work with.

Because of the relation given by the Fundamental Theorem between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of *f* and is called an **indefinite integral.**

Indefinite Integrals

Thus

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

So we can regard an indefinite integral as representing an entire *family* of functions (one antiderivative for each value of the constant *C*).

Indefinite Integrals

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*, whereas an indefinite integral $\int f(x)\,dx$ is a *function* (or family of functions).

The connection between them is given by Part 2 of the Fundamental Theorem:

If *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b$$

The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions.

Indefinite Integrals

Any formula can be verified by differentiating the function on the right side and obtaining the integrand.

For instance,

$$\int \sec^2 x \, dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

Indefinite Integrals

1 Table of Indefinite Integrals

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	
$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int a^x\,dx = \dfrac{a^x}{\ln a} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \dfrac{1}{x^2+1}\,dx = \tan^{-1}x + C$	$\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$
$\int \sinh x\,dx = \cosh x + C$	$\int \cosh x\,dx = \sinh x + C$

The most general antiderivative *on a given interval* is obtained by adding a constant to a particular antiderivative.

Indefinite Integrals

We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval.

Thus we write

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$.

Indefinite Integrals

This is true despite the fact that the general antiderivative of the function $f(x) = 1/x^2$, $x \neq 0$, is

$$F(x) = \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x < 0 \\ -\dfrac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

Example 2

Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

Solution:

This indefinite integral isn't immediately apparent in Table 1, so we use trigonometric identities to rewrite the function before integrating:

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$= \int \csc\theta \cot\theta \, d\theta$$

$$= -\csc\theta + C$$

Example 5

Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt.$

Solution:

First we need to write the integrand in a simpler form by carrying out the division:

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt = \int_1^9 (2 + t^{1/2} - t^{-2})\, dt$$

$$= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\Bigg]_1^9$$

$$= 2t + \tfrac{2}{3}t^{3/2} + \frac{1}{t}\Bigg]_1^9$$

Example 5 – *Solution*

cont'd

$$= \left(2 \cdot 9 + \tfrac{2}{3} \cdot 9^{3/2} + \tfrac{1}{9}\right) - \left(2 \cdot 1 + \tfrac{2}{3} \cdot 1^{3/2} + \tfrac{1}{1}\right)$$

$$= 18 + 18 + \tfrac{1}{9} - 2 - \tfrac{2}{3} - 1$$

$$= 32\tfrac{4}{9}$$

Applications

Applications

Part 2 of the Fundamental Theorem says that if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f. This means that $F' = f$, so the equation can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Applications

We know that $F'(x)$ represents the rate of change of $y = F(x)$

with respect to x and $F(b) - F(a)$ is the change in y when x changes from a to b.

[Note that y could, for instance, increase, then decrease, then increase again.

Although y might change in both directions, $F(b) - F(a)$ represents the *net* change in y.]

Applications

So we can reformulate FTC2 in words as follows.

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This principle can be applied to all of the rates of change in the natural and social sciences. Here are a few instances of this idea:

- If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t.

Applications

So

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time t_1 and time t_2.

- If [C](t) is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $d[C]/dt$.

Applications

So

$$\int_{t_1}^{t_2} \frac{d[\mathrm{C}]}{dt}\, dt = [\mathrm{C}](t_2) - [\mathrm{C}](t_1)$$

is the change in the concentration of C from time t_1 to time t_2.

- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\, dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between

$x = a$ and $x = b$.

Applications

- If the rate of growth of a population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

is the net change in population during the time period from t_1 to t_2.

(The population increases when births happen and decreases when deaths occur. The net change takes into

account both births and deaths.)

Applications

- If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$.

  So

$$\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$$

  is the increase in cost when production is increased from x_1 units to x_2 units.

Applications

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

2
$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2.

This was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

Applications

- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left).

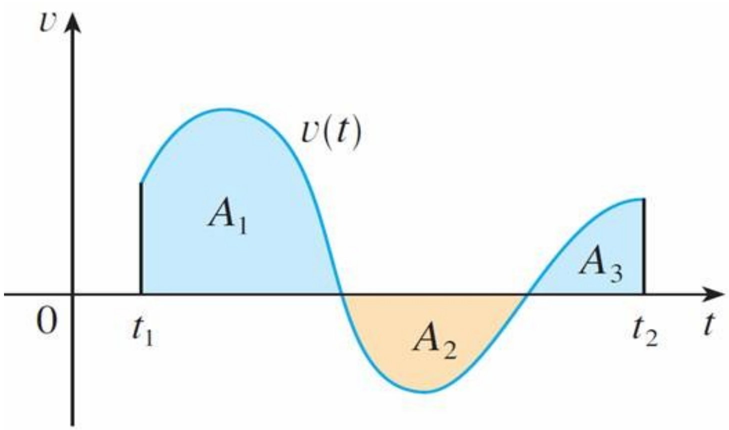
  In both cases the distance is computed by integrating $|v(t)|$,

  the speed. Therefore

3 $$\int_{t_1}^{t_2} |v(t)|\,dt = \text{total distance traveled}$$

Applications

Figure 3 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.



$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

Figure 3

Applications

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

is the change in velocity from time t_1 to time t_2.

Example 6

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during this time period.

Example 6 – *Solution*

(a) By Equation 2, the displacement is

$$s(4) - s(1) = \int_1^4 v(t)\,dt$$

$$= \int_1^4 (t^2 - t - 6)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4$$

$$= -\frac{9}{2}$$

Example 6 – *Solution*

cont'd

This means that the particle moved 4.5 m toward the left.

(b) Note that $v(t) = t^2 - t - 6 = (t - 3)(t + 2)$ and so $v(t) \leq 0$ on the interval [1, 3] and $v(t) \geq 0$ on [3, 4].

Thus, from Equation 3, the distance traveled is

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$

Example 6 – *Solution*

cont'd

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t \right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^4$$

$$= \frac{61}{6}$$

$$\approx 10.17 \text{ m}$$