

6

Applications of Integration



6.1

More About Areas

Areas Between Curves

Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$. (See Figure 1.)

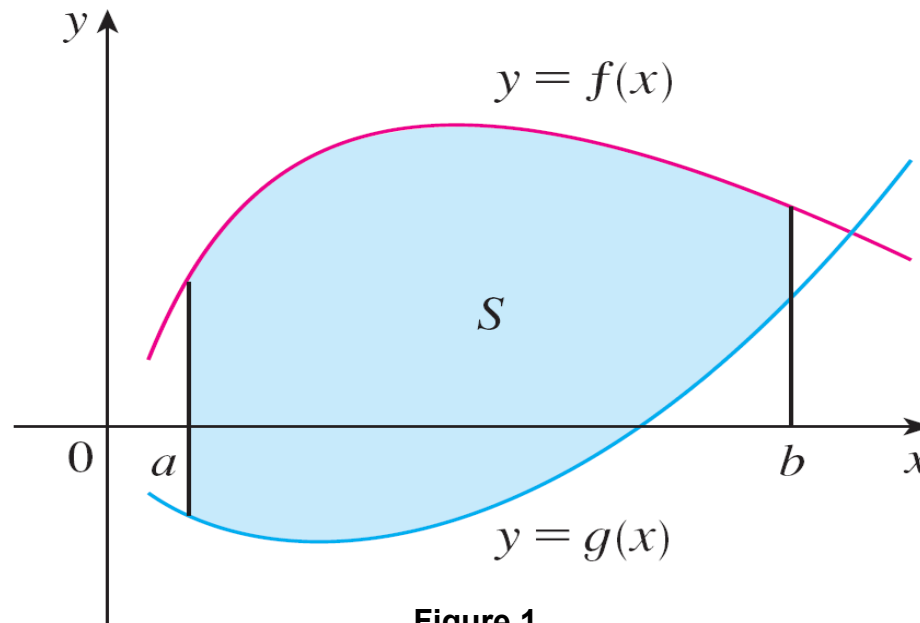
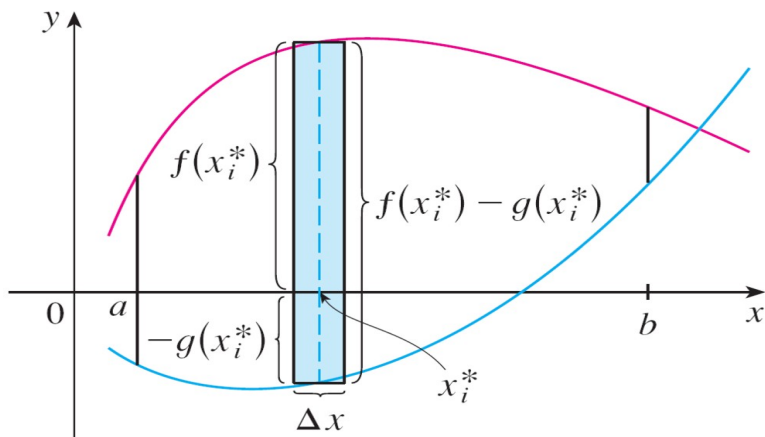


Figure 1

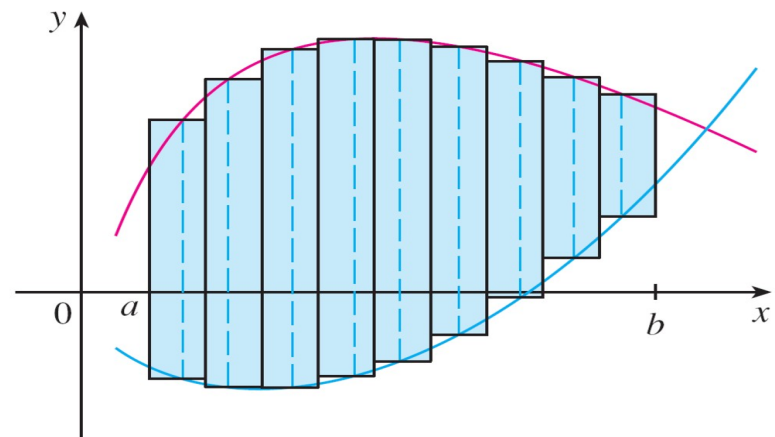
$$S = \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$$

Areas Between Curves

We divide S into n strips of equal width and then we approximate the i th strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$. (See Figure 2. If we like, we could take all of the sample points to be right endpoints, in which case $x_i^* = x_i$.)



(a) Typical rectangle



(b) Approximating rectangles

Figure 2

Areas Between Curves

The Riemann sum

$$\sum_{i=1}^n [f(x_i^*) - g(x_i^*)] \Delta x$$

is therefore an approximation to what we intuitively think of as the area of S .

This approximation appears to become better and better as $n \rightarrow \infty$. Therefore we define the **area** A of the region S as the limiting value of the sum of the areas of these approximating rectangles.

Areas Between Curves

1

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n [f(x_i^*) - g(x_i^*)] \Delta x$$

We recognize the limit in **1** as the definite integral of $f - g$. Therefore we have the following formula for area.

2 The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)] dx$$

Notice that in the special case where $g(x) = 0$, S is the region under the graph of f and our general definition of area **1** reduces.

Areas Between Curves

In the case where both f and g are positive, you can see from Figure 3 why [2](#) is true:

$$\begin{aligned} A &= [\text{area under } y = f(x)] - [\text{area under } y = g(x)] \\ &= \int_a^b f(x) dx - \int_a^b g(x) dx = \int_a^b [f(x) - g(x)] dx \end{aligned}$$

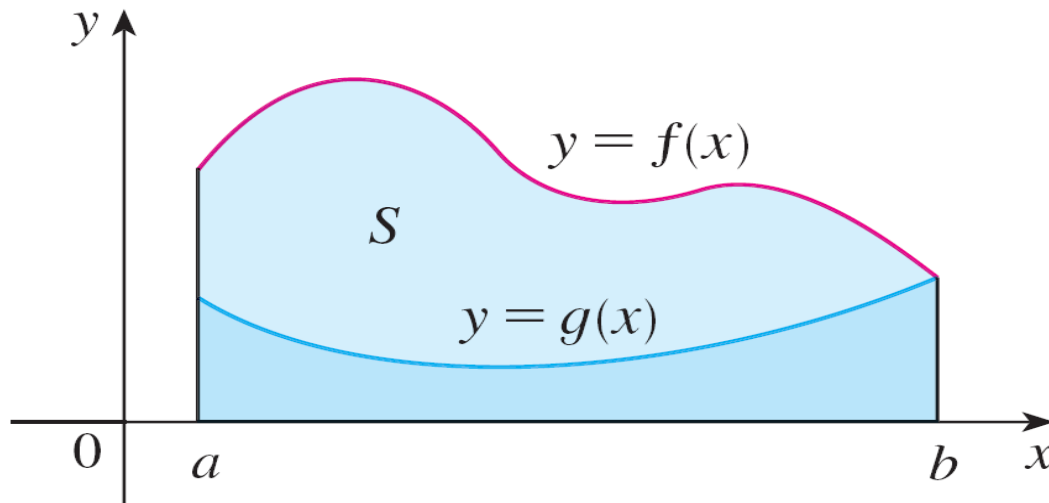


Figure 3

$$A = \int_a^b f(x) dx - \int_a^b g(x) dx$$

Example 1

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Solution:

The region is shown in Figure 4. The upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$.

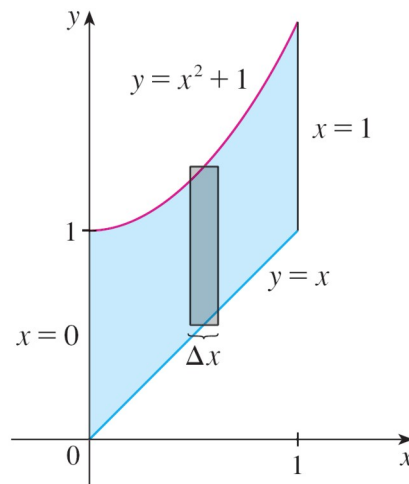


Figure 4

Example 1 – Solution

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So we use the area formula $\boxed{2}$ with $f(x) = e^x$, $g(x) = x$, $a = 0$, and $b = 1$:

$$\begin{aligned} A &= \int_0^1 (e^x - x) dx = e^x - \frac{1}{2}x^2 \Big|_0^1 \\ &= e - \frac{1}{2} - 1 = e - 1.5 \end{aligned}$$

Areas Between Curves

In Figure 4 we drew a typical approximating rectangle with width Δx as a reminder of the procedure by which the area is defined in [1](#).

In general, when we set up an integral for an area, it's helpful to sketch the region to identify the top curve y_T , the bottom curve y_B , and a typical approximating rectangle as in Figure 5.

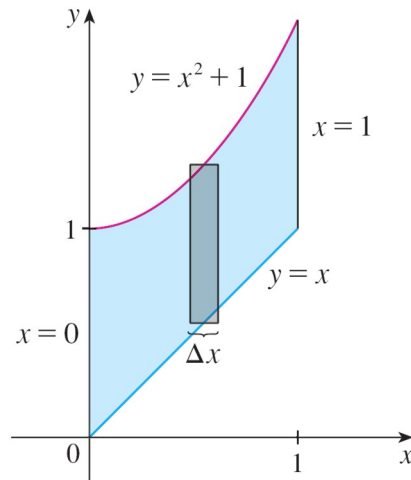


Figure 4

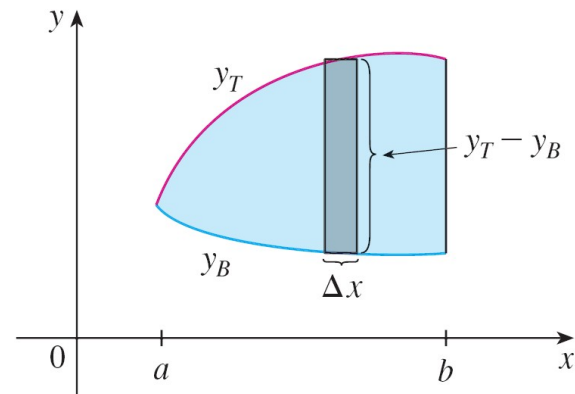


Figure 5

Areas Between Curves

Then the area of a typical rectangle is $(y_T - y_B) \Delta x$ and the equation

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n (y_T - y_B) \Delta x = \int_a^b (y_T - y_B) dx$$

summarizes the procedure of adding (in a limiting sense) the areas of all the typical rectangles.

Notice that in Figure 5 the left-hand boundary reduces to a point, whereas in Figure 3 the right-hand boundary reduces to a point.

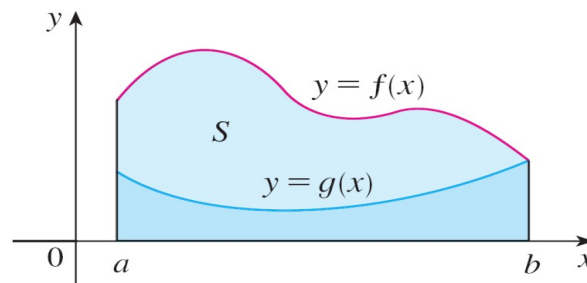


Figure 3

$$A = \int_a^b f(x) dx - \int_a^b g(x) dx$$

Areas Between Curves

If we are asked to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ for some values x of but $g(x) \geq f(x)$ for other values of x , then we split

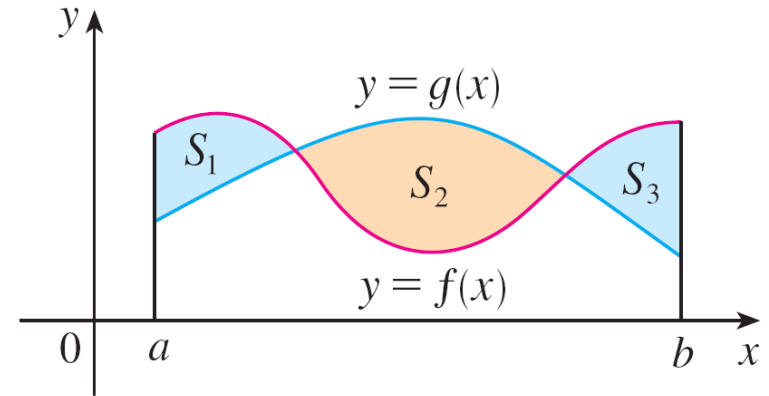


Figure 9

the given region S into several regions S_1, S_2, \dots with areas A_1, A_2, \dots as shown in Figure 9. We then define the area of the region S to be the sum of the areas of the smaller regions S_1, S_2, \dots that is $A = A_1 + A_2 + \dots$. Since

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } g(x) \geq f(x) \end{cases}$$

Areas Between Curves

we have the following expression for A .

3 The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)| dx$$

When evaluating the integral in **3**, however, we must still split it into integrals corresponding to A_1, A_2, \dots

Example 5

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$

Solution:

The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ (since $0 \leq x \leq \pi/2$). The region is sketched in Figure 10. Observe that $\cos x \geq \sin x$ when $0 \leq x \leq \pi/4$ but $\sin x \geq \cos x$ when $\pi/4 \leq x \leq \pi/2$.

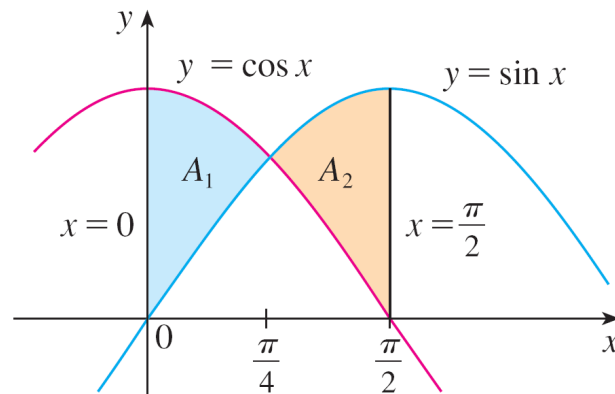


Figure 10

Example 5 – Solution

cont'd

Therefore the required area is

$$\begin{aligned} A &= \int_0^{\pi/2} |\cos x - \sin x| dx = A_1 + A_2 \\ &= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx \\ &= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2} \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 \right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \\ &= 2\sqrt{2} - 2 \end{aligned}$$

Example 5 – *Solution*

cont'd

In this particular example we could have saved some work by noticing that the region is symmetric about $x = \pi/4$ and so

$$A = 2A_1 = 2 \int_0^{\pi/4} (\cos x - \sin x) dx$$

Areas Between Curves

Some regions are best treated by regarding x as a function of y . If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ (see Figure 11), then its area is

$$A = \int_c^d [f(y) - g(y)] dy$$

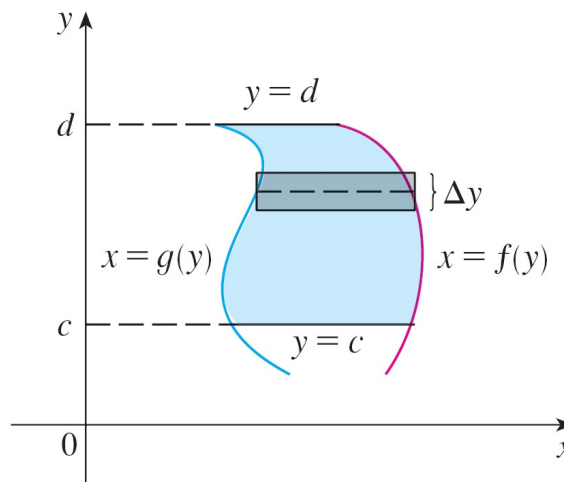


Figure 11

Areas Between Curves

If we write x_R for the right boundary and x_L for the left boundary, then, as Figure 12 illustrates, we have

$$A = \int_c^d (x_R - x_L) dy$$

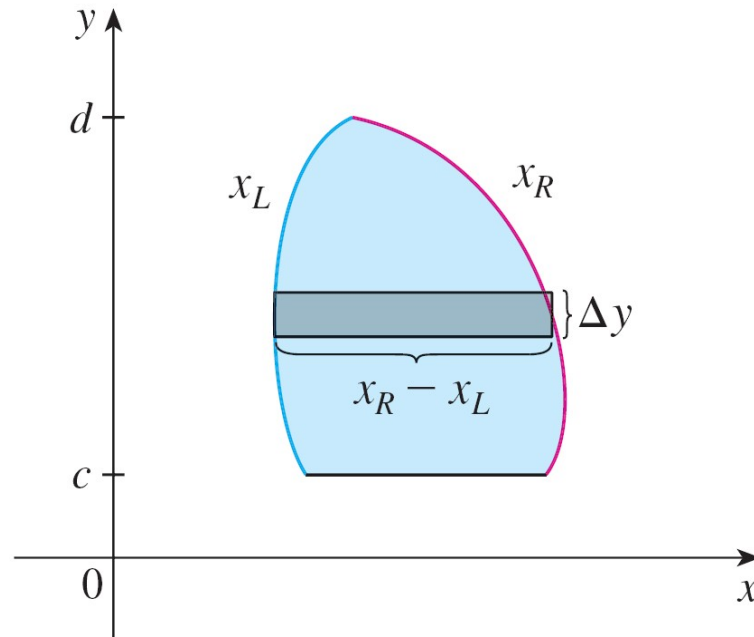


Figure 12