6

Applications of Integration

6.4 Work

Work

The term *work* is used in everyday language to mean the total amount of effort required to perform a task.

In physics it has a technical meaning that depends on the idea of a *force.*

Intuitively, you can think of a force as describing a push or pull on an object—for example, a horizontal push of a book across a table or the downward pull of the earth’s gravity on a ball.

Work

In general, if an object moves along a straight line with position function *s*(*t*), then the **force** *F* on the object (in the same direction) is given by Newton's Second Law of Motion as the product of its mass *m* and its acceleration:

$$F = m\frac{d^2s}{dt^2} \qquad (1)$$

In the SI metric system, the mass is measured in kilograms (kg), the displacement in meters (m), the time in seconds (s), and the force in newtons (N = kg•m/s^2). Thus a force of 1 N acting on a mass of 1 kg produces an acceleration of 1 m/s^2.

Work

In the US Customary system the fundamental unit is chosen to be the unit of force, which is the pound.

In the case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and the distance d that the object moves:

2 $$W = Fd \qquad \text{work} = \text{force} \times \text{distance}$$

Work

If F is measured in newtons and d in meters, then the unit for W is a newton-meter, which is called a joule (J).

If F is measured in pounds and d in feet, then the unit for W is a foot-pound (ft-lb), which is about 1.36 J.

Example 1

(a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is $g = 9.8 \text{ m/s}^2$.

(b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

Example 1(a) – *Solution*

The force exerted is equal and opposite to that exerted by gravity, so Equation 1 gives

$$F = mg = (1.2)(9.8)$$

$$= 11.76 \text{ N}$$

and then Equation 2 gives the work done as

$$W = Fd = (11.76)(0.7)$$

$$\approx 8.2 \text{ J}$$

Example 1(b) – *Solution*

cont'd

Here the force is given as F = 20 lb, so the work done is

$$W = Fd = 20 \cdot 6$$

$$= 120 \text{ ft-lb}$$

Notice that in part (b), unlike part (a), we did not have to multiply by g because we were given the *weight* (which is a force) and not the mass of the object.

Work

Equation 2 defines work as long as the force is constant, but what happens if the force is variable? Let's suppose that the object moves along the x-axis in the positive direction, from $x = a$ to $x = b$, and at each point x between a and b a force $f(x)$ acts on the object, where f is a continuous function.

We divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx.

We choose a sample point x_i^* in the ith subinterval $[x_{i-1}, x_i]$. Then the force at that point is $f(x_i^*)$.

Work

If n is large, then Δx is small, and since f is continuous, the values of f don't change very much over the interval $[x_{i-1}, x_i]$.

In other words, f is almost constant on the interval and so the work W_i that is done in moving the particle from x_{i-1} to x_i is approximately given by Equation 2:

$$W_i \approx f(x_i^*)\,\Delta x$$

Thus we can approximate the total work by

3 $$W \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Work

It seems that this approximation becomes better as we make *n* larger. Therefore we define the **work done in moving the object from *a* to *b*** as the limit of this quantity as $n \to \infty$.

Since the right side of $\boxed{3}$ is a Riemann sum, we recognize its limit as being a definite integral and so

$$\boxed{4} \qquad W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x = \int_a^b f(x)\, dx$$

Example 2

When a particle is located a distance *x* feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

Solution:

$$W = \int_1^3 (x^2 + 2x)\, dx = \frac{x^3}{3} + x^2 \Big]_1^3$$

$$= \frac{50}{3}$$

The work done is $16\frac{2}{3}$ ft-lb.

Work

In the next example we use a law from physics:

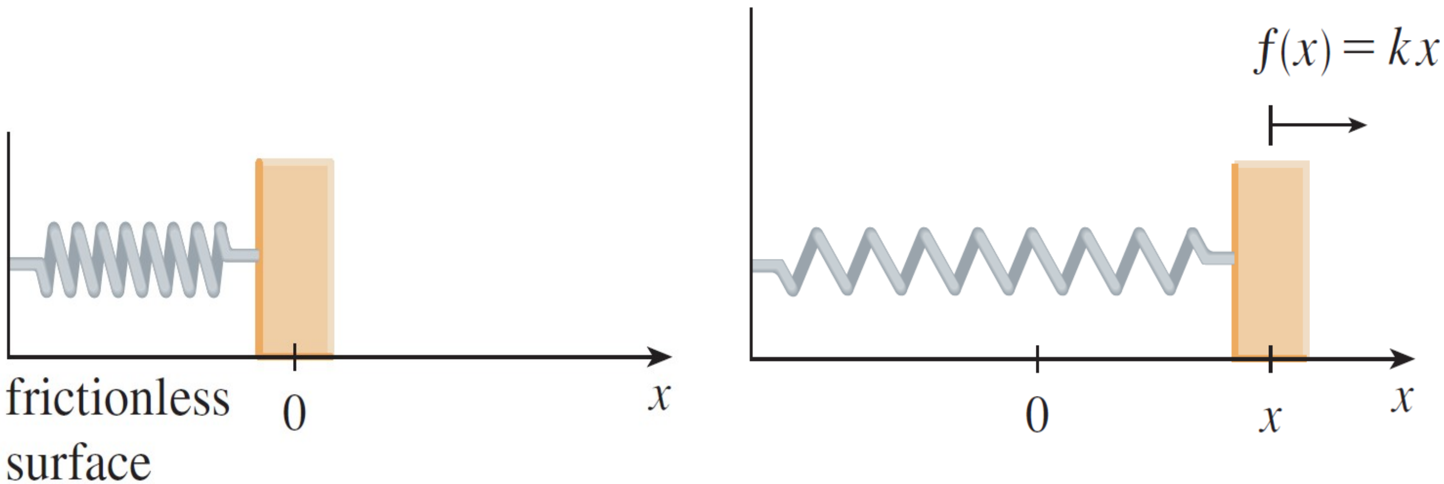
Hooke's Law states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$$f(x) = kx$$

where k is a positive constant (called the **spring constant**).

Work

Hooke's Law holds provided that *x* is not too large (see Figure 1).



(a) Natural position of spring

(b) Stretched position of spring

Hooke's Law

Figure 1

Example 3

A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

Solution:

According to Hooke's Law, the force required to hold the spring stretched x meters beyond its natural length is $f(x) = kx$.

Example 3 – *Solution*

cont'd

When the spring is stretched from 10 cm to 15 cm, the amount stretched is 5 cm = 0.05 m. This means that $f(0.05) = 40$, so

$$0.05k = 40 \qquad k = \frac{40}{0.05} = 800$$

Thus $f(x) = 800x$ and the work done in stretching the spring from 15 cm to 18 cm is

$$W = \int_{0.05}^{0.08} 800x\,dx = 800 \frac{x^2}{2}\Bigg]_{0.05}^{0.08}$$

$$= 400[(0.08)^2 – 0.05)^2]$$

$$= 1.56 \text{ J}$$