7

Techniques of Integration

7.3 Trigonometric Substitution

Trigonometric Substitution

In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\, dx$ arises, where $a > 0$.

If it were $\int x\sqrt{a^2 - x^2}\, dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\, dx$ is more difficult.

Trigonometric Substitution

If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta}$$

$$= \sqrt{a^2(1 - \sin^2\theta)}$$

$$= \sqrt{a^2 \cos^2\theta}$$

$$= a\,|\cos \theta|$$

Trigonometric Substitution

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution $x = a \sin\theta$ (the old variable is a function of the new one).

In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse.

To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one.

Trigonometric Substitution

In this case, if we replace *u* by *x* and *x* by *t* in the Substitution Rule, we obtain

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt$$

This kind of substitution is called *inverse substitution*.

We can make the inverse substitution $x = a \sin \theta$ provided that it defines a one-to-one function.

Trigonometric Substitution

This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Trigonometric Substitution

In each case the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one.

Example 1

Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\,dx.$

Solution:

Let $x = 3 \sin \theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 3 \cos \theta \, d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2\theta}$$

$$= \sqrt{9 \cos^2\theta}$$

$$= 3\,|\cos \theta|$$

$$= 3 \cos \theta$$

(Note that $\cos \theta \ge 0$ because $-\pi/2 \le \theta \le \pi/2$.)

Example 1 – *Solution*

cont'd

Thus the Inverse Substitution Rule gives

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\,d\theta$$

$$= \int (\csc^2\theta - 1)\,d\theta$$

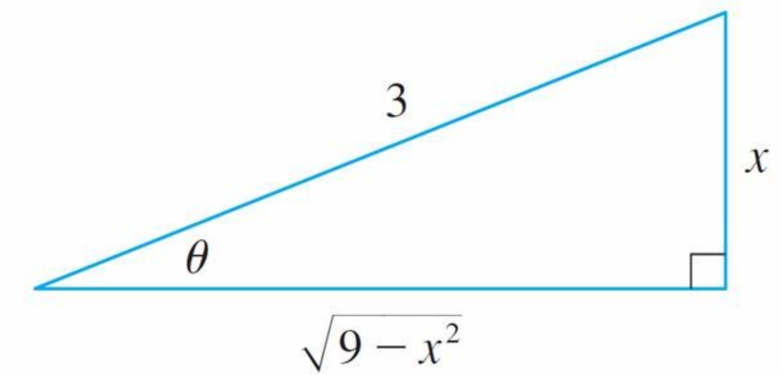
$$= -\cot\theta - \theta + C$$

Example 1 – *Solution*

cont'd

Since this is an indefinite integral, we must return to the original variable *x*.

This can be done either by using trigonometric identities to express cot θ in terms of sin $\theta = x/3$ or by drawing a diagram, as in Figure 1, where θ is interpreted as an angle of a right triangle.



$\sin\theta = \frac{x}{3}$

Figure 1

Example 1 – *Solution*

Since sin $\theta = x/3$, we label the opposite side and the hypotenuse as having lengths x and 3.

Then the Pythagorean Theorem gives the length of the adjacent side as $\sqrt{9 - x^2}$, so we can simply read the value of cot θ from the figure:

$$\cot \theta = \frac{\sqrt{9 - x^2}}{x}$$

(Although $\theta > 0$ in the diagram, this expression for cot θ is valid even when $\theta < 0$.)

Example 1 – *Solution*

cont'd

Since sin $\theta = x/3$, we have $\theta = \sin^{-1}(x/3)$ and so

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx = -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

Example 2

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution:

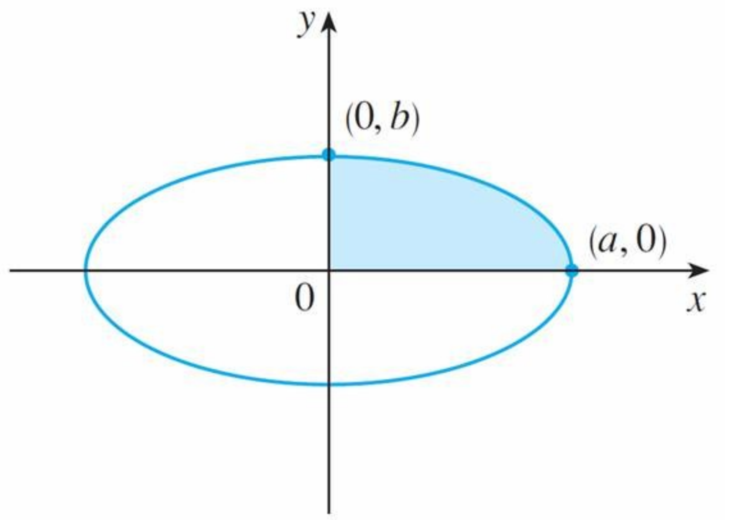
Solving the equation of the ellipse for *y*, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \qquad \text{or} \qquad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

Example 2 – *Solution*

cont'd

Because the ellipse is symmetric with respect to both axes, the total area *A* is four times the area in the first quadrant (see Figure 2).



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Figure 2

Example 2 – *Solution*

cont'd

The part of the ellipse in the first quadrant is given by the function

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \leqslant x \leqslant a$$

and so

$$\tfrac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

To evaluate this integral we substitute $x = a \sin \theta$.
Then $dx = a \cos \theta \, d\theta$.

Example 2 – *Solution*

cont'd

To change the limits of integration we note that when $x = 0$, $\sin\theta = 0$, so $\theta = 0$; when $x = a$, $\sin\theta = 1$, so $\theta = \pi/2$. Also

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta}$$

$$= \sqrt{a^2 \cos^2\theta}$$

$$= a\left|\cos\theta\right|$$

$$= a\cos\theta$$

since $0 \le \theta \le \pi/2$.

Example 2 – *Solution*

cont'd

Therefore

$$A = 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta)\,d\theta$$

Example 2 – *Solution*

cont'd

$$= 2ab\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= 2ab\left(\frac{\pi}{2} + 0 - 0\right)$$

$$= \pi ab$$

We have shown that the area of an ellipse with semiaxes *a* and *b* is πab.

In particular, taking $a = b = r$, we have proved the famous formula that the area of a circle with radius *r* is πr^2.

Trigonometric Substitution

Note:

Since the integral in Example 2 was a definite integral, we changed the limits of integration and did not have to convert back to the original variable x.

Example 3

Find $\int \frac{1}{x^2\sqrt{x^2 + 4}}\, dx$.

Solution:

Let $x = 2 \tan \theta$, $-\pi/2 < \theta < \pi/2$. Then $dx = 2 \sec^2\theta \, d\theta$ and

$$\sqrt{x^2 + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4 \sec^2\theta}$$

$$= 2|\sec \theta|$$

$$= 2 \sec \theta$$

Example 3 – *Solution*

cont'd

Thus we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta \cdot 2\sec\theta}$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

To evaluate this trigonometric integral we put everything in terms of sin θ and cos θ :

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

Example 3 – *Solution*

cont'd

Therefore, making the substitution $u = \sin\,\theta$, we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \frac{1}{4}\int \frac{\cos\,\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{du}{u^2}$$

$$= \frac{1}{4}\left(-\frac{1}{u}\right) + C$$

$$= -\frac{1}{4\,\sin\,\theta} + C$$

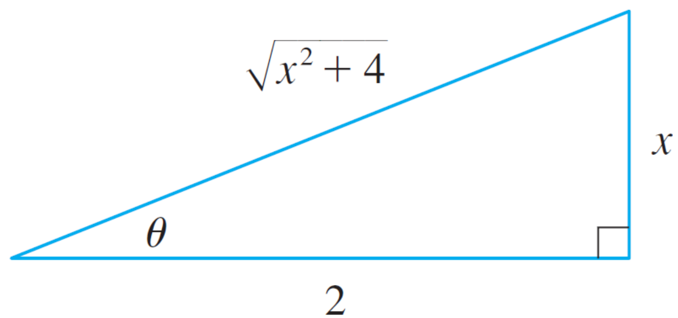
$$= -\frac{\csc\,\theta}{4} + C$$

Example 3 – *Solution*

cont'd

We use Figure 3 to determine that csc $\theta = \sqrt{x^2 + 4}/x$ and so

$$\int \frac{dx}{x^2\sqrt{x^2 + 4}} = -\frac{\sqrt{x^2 + 4}}{4x} + C$$



$\tan \theta = \frac{x}{2}$

Figure 3

Example 5

Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$.

Solution 1:

We let $x = a \sec \theta$, where $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$.

Then $dx = a \sec \theta \tan \theta \, d\theta$ and

$$\sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)}$$

$$= \sqrt{a^2 \tan^2\theta}$$

$$= a\left|\tan \theta\right|$$

$$= a \tan \theta$$

Example 5 – *Solution* 1

cont'd

Therefore

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec \theta \tan \theta}{a \tan \theta} d\theta$$

$$= \int \sec \theta \, d\theta$$

$$= \ln |\sec \theta + \tan \theta| + C$$

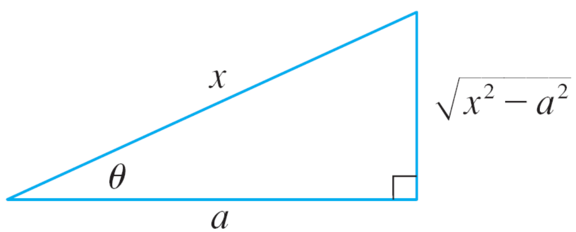
Example 5 – *Solution* 1

cont'd

The triangle in Figure 4 gives $\tan\theta = \sqrt{x^2 - a^2}/a$, so we have

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C$$

$$= \ln\left|x + \sqrt{x^2 - a^2}\right| - \ln a + C$$



$$\sec\theta = \frac{x}{a}$$

Figure 4

Example 5 – *Solution* 1

cont'd

Writing $C_1 = C - \ln a$, we have

1

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + C_1$$

Example 5 – *Solution 2*

cont'd

For $x > 0$ the hyperbolic substitution $x = a \cosh t$ can also be used.

Using the identity $\cosh^2 y - \sinh^2 y = 1$, we have

$$\sqrt{x^2 - a^2} = \sqrt{a^2(\cosh^2 t - 1)}$$

$$= \sqrt{a^2 \sinh^2 t}$$

$$= a \sinh t$$

Example 5 – *Solution* 2

cont'd

Since $dx = a \sinh t\, dt$, we obtain

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sinh t\, dt}{a \sinh t}$$

$$= \int dt$$

$$= t + C$$

Since $\cosh t = x/a$, we have $t = \cosh^{-1}(x/a)$ and

2 $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

Trigonometric Substitution

Note:
As Example 5 illustrates, hyperbolic substitutions can be used in place of trigonometric substitutions and sometimes they lead to simpler answers.

But we usually use trigonometric substitutions because trigonometric identities are more familiar than hyperbolic identities.

Example 6

Find $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx.$

Solution:

First we note that $(4x^2 + 9)^{3/2} = (\sqrt{4x^2+9})^3$ so trigonometric substitution is appropriate.

Although $\sqrt{4x^2+9}$ is not quite one of the expressions in the table of trigonometric substitutions, it becomes one of them if we make the preliminary substitution $u = 2x$.

Example 6 – *Solution*

cont'd

When we combine this with the tangent substitution, we have $x = \frac{3}{2}\tan\theta$, which gives $dx = \frac{3}{2}\sec^2\theta\, d\theta$ and

$$\sqrt{4x^2 + 9} = \sqrt{9\tan^2\theta + 9}$$

$$= 3\sec\theta$$

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$; when $x = 3\sqrt{3}/2$, $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$.

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\, dx = \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\, \frac{3}{2}\sec^2\theta\, d\theta$$

Example 6 – *Solution*

cont'd

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\, d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\, d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{1 - \cos^2\theta}{\cos^2\theta} \sin\theta\, d\theta$$

Now we substitute $u = \cos\theta$ so that $du = -\sin\theta\, d\theta$.
When $\theta = 0$, $u = 1$; when $\theta = \pi/3$, $u = \frac{1}{2}$.

Example 6 – *Solution*

cont'd

Therefore

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = -\frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\,du$$

$$= \frac{3}{16}\int_1^{1/2} (1-u^{-2})\,du$$

$$= \frac{3}{16}\left[u+\frac{1}{u}\right]_1^{1/2}$$

$$= \frac{3}{16}\left[\left(\tfrac{1}{2}+2\right)-(1+1)\right]$$

$$= \frac{3}{32}$$