

7

Techniques of Integration



7.2

Trigonometric Integrals

Trigonometric Integrals

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions.

We start with powers of sine and cosine.

Example 2

Find $\int \sin^5 x \cos^2 x \, dx$.

Solution:

We could convert $\cos^2 x$ to $1 - \sin^2 x$, but we would be left with an expression in terms of $\sin x$ with no extra $\cos x$ factor.

Instead, we separate a single sine factor and rewrite the remaining $\sin^4 x$ factor in terms of $\cos x$:

$$\begin{aligned}\sin^5 x \cos^2 x &= (\sin^2 x)^2 \cos^2 x \sin x \\ &= (1 - \cos^2 x)^2 \cos^2 x \sin x\end{aligned}$$

Example 2 – Solution

cont'd

Substituting $u = \cos x$, we have $du = -\sin x dx$ and so

$$\begin{aligned}\int \sin^5 x \cos^2 x dx &= \int (\sin^2 x)^2 \cos^2 x \sin x dx \\ &= \int (1 - \cos^2 x)^2 \cos^2 x \sin x dx \\ &= \int (1 - u^2)^2 u^2 (-du) = -\int (u^2 - 2u^4 + u^6) du \\ &= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C \\ &= -\frac{1}{3} \cos^3 x + \frac{2}{5} \cos^5 x - \frac{1}{7} \cos^7 x + C\end{aligned}$$

Example 3

Evaluate $\int_0^{\pi} \sin^2 x \, dx$.

Solution:

If we write $\sin^2 x = 1 - \cos^2 x$, the integral is no simpler to evaluate. Using the half-angle formula for $\sin^2 x$, however, we have

$$\begin{aligned} \int_0^{\pi} \sin^2 x \, dx &= \frac{1}{2} \int_0^{\pi} (1 - \cos 2x) \, dx \\ &= \left[\frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) \right]_0^{\pi} \end{aligned}$$

Example 3 – *Solution*

cont'd

$$\begin{aligned} &= \frac{1}{2}(\pi - \frac{1}{2} \sin 2\pi) - \frac{1}{2}(0 - \frac{1}{2} \sin 0) \\ &= \frac{1}{2} \pi \end{aligned}$$

Notice that we mentally made the substitution $u = 2x$ when integrating $\cos 2x$.

Trigonometric Integrals

To summarize, we list guidelines to follow when evaluating integrals of the form $\int \sin^m x \cos^n x dx$, where $m \geq 0$ and $n \geq 0$ are integers.

Trigonometric Integrals

Strategy for Evaluating $\int \sin^m x \cos^n x dx$

- (a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x dx &= \int \sin^m x (\cos^2 x)^k \cos x dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x dx\end{aligned}$$

Then substitute $u = \sin x$.

- (b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x dx &= \int (\sin^2 x)^k \cos^n x \sin x dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x dx\end{aligned}$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

- (c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Trigonometric Integrals

We can use a similar strategy to evaluate integrals of the form $\int \tan^m x \sec^n x dx$.

Since $(d/dx) \tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

Or, since $(d/dx) \sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining (even) power of tangent to secant.

Example 5

Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Solution:

If we separate one $\sec^2 x$ factor, we can express the remaining $\sec^2 x$ factor in terms of tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

We can then evaluate the integral by substituting $u = \tan x$ so that $du = \sec^2 x \, dx$:

$$\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \sec^2 x \sec^2 x \, dx$$

Example 5 – Solution

cont'd

$$= \int \tan^6 x (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int u^6 (1 + u^2) du = \int (u^6 + u^8) du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{1}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C$$

Trigonometric Integrals

The preceding examples demonstrate strategies for evaluating integrals of the form $\int \tan^m x \sec^n x dx$ for two cases, which we summarize here.

Strategy for Evaluating $\int \tan^m x \sec^n x dx$

- (a) If the power of secant is even ($n = 2k, k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\begin{aligned}\int \tan^m x \sec^{2k} x dx &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x dx \\ &= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x dx\end{aligned}$$

Then substitute $u = \tan x$.

- (b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x dx\end{aligned}$$

Then substitute $u = \sec x$.

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For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity.

We will sometimes need to be able to integrate $\tan x$ by using the formula given below:

$$\int \tan x \, dx = \ln |\sec x| + C$$

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We will also need the indefinite integral of secant:

$$1 \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

We could verify Formula 1 by differentiating the right side, or as follows. First we multiply numerator and denominator by $\sec x + \tan x$:

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

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$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

If we substitute $u = \sec x + \tan x$, then $du = (\sec x \tan x + \sec^2 x)dx$, so the integral becomes

$$\int (1/u) du = \ln |u| + C.$$

Thus we have

$$\int \sec x dx = \ln |\sec x + \tan x| + C$$

Example 7

Find $\int \tan^3 x \, dx$.

Solution:

Here only $\tan x$ occurs, so we use $\tan^2 x = \sec^2 x - 1$ to rewrite a $\tan^2 x$ factor in terms of $\sec^2 x$:

$$\begin{aligned}\int \tan^3 x \, dx &= \int \tan x \tan^2 x \, dx \\ &= \int \tan x (\sec^2 x - 1) \, dx \\ &= \int \tan x \sec^2 x \, dx - \int \tan x \, dx\end{aligned}$$

Example 7 – Solution

cont'd

$$= \frac{\tan^2 x}{2} - \ln |\sec x| + C$$

In the first integral we mentally substituted $u = \tan x$ so that $du = \sec^2 x dx$.

Trigonometric Integrals

Finally, we can make use of another set of trigonometric identities:

2 To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

$$(a) \quad \sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$(b) \quad \sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$(c) \quad \cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Example 9

Evaluate $\int \sin 4x \cos 5x \, dx$.

Solution:

This integral could be evaluated using integration by parts, but it's easier to use the identity in Equation 2(a) as follows:

$$\begin{aligned}\int \sin 4x \cos 5x \, dx &= \int \frac{1}{2} [\sin(-x) + \sin 9x] \, dx \\ &= \frac{1}{2} \int (-\sin x + \sin 9x) \, dx \\ &= \frac{1}{2} \left(\cos x - \frac{1}{9} \cos 9x \right) + C\end{aligned}$$