7

Techniques of Integration

7.4 Integration of Rational Functions by Partial Fractions

Integration of Rational Functions by Partial Fractions

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions*, that we already know how to integrate.

To illustrate the method, observe that by taking the fractions $2/(x - 1)$ and $1/(x + 2)$ to a common denominator we obtain

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

Integration of Rational Functions by Partial Fractions

If we now reverse the procedure, we see how to integrate the function on the right side of this equation:

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$

$$= 2 \ln |x - 1| - \ln |x + 2| + C$$

Integration of Rational Functions by Partial Fractions

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where *P* and *Q* are polynomials. It's possible to express *f* as a sum of simpler fractions provided that the degree of *P* is less than the degree of *Q*. Such a rational function is called *proper*.

Integration of Rational Functions by Partial Fractions

Recall that if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, then the degree of P is n and we write $\deg(P) = n$.

If f is *improper*, that is, $\deg(P) \geq \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$.

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The division statement is

1 $$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

As the next example illustrates, sometimes this preliminary step is all that is required.

Example 1

Find $\int \frac{x^3 + x}{x - 1}\, dx$.

Solution:

Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division.

This enables us to write

$$\int \frac{x^3 + x}{x - 1}\, dx = \int \left(x^2 + x + 2 + \frac{2}{x - 1} \right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln |x - 1| + C$$

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The next step is to factor the denominator $Q(x)$ as far as possible.

It can be shown that any polynomial Q can be factored as a product of linear factors (of the form $ax + b$) and irreducible quadratic factors (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

For instance, if $Q(x) = x^4 - 16$, we could factor it as

$$Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

Integration of Rational Functions by Partial Fractions

The third step is to express the proper rational function $R(x)/Q(x)$ (from Equation 1) as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

Case I: The denominator *Q*(*x*) is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another).

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In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\boxed{2} \qquad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

These constants can be determined as in the next example.

Example 2

Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx.$

Solution:

Since the degree of the numerator is less than the degree of the denominator, we don't need to divide.

We factor the denominator as

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2)$$

$$= x(2x - 1)(x + 2)$$

Example 2 – *Solution*

cont'd

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand [2] has the form

$$\boxed{3} \qquad \frac{x^2 + 2x - 1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

To determine the values of *A*, *B*, and *C*, we multiply both sides of this equation by the product of the denominators, $x(2x-1)(x+2)$, obtaining

$$\boxed{4} \qquad x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

Example 2 – *Solution*

cont'd

Expanding the right side of Equation 4 and writing it in the standard form for polynomials, we get

$$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A \qquad \boxed{5}$$

The polynomials in Equation 5 are identical, so their coefficients must be equal. The coefficient of x^2 on the right side, $2A + B + 2C$, must equal the coefficient of x^2 on the left side—namely, 1.

Likewise, the coefficients of x are equal and the constant terms are equal.

Example 2 – *Solution*

cont'd

This gives the following system of equations for A, B, and C:

$$2A + B + 2C = 1$$

$$3A + 2B - C = 2$$

$$-2A = -1$$

Solving, we get, $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$, and so

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx = \int \left[\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x - 1} - \frac{1}{10}\frac{1}{x + 2}\right] dx$$

Example 2 – *Solution*

cont'd

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x - 1| - \frac{1}{10}\ln|x + 2| + K$$

In integrating the middle term we have made the mental substitution $u = 2x - 1$, which gives $du = 2dx$ and $dx = \frac{1}{2}du$.

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Note:
We can use an alternative method to find the coefficients *A*, *B* and *C* in Example 2. Equation 4 is an identity; it is true for every value of *x*. Let's choose values of *x* that simplify the equation.

If we put $x = 0$ in Equation 4, then the second and third terms on the right side vanish and the equation then becomes $-2A = -1$, or $A = \frac{1}{2}$.

Likewise, $x = \frac{1}{2}$ gives $5B/4 = \frac{1}{4}$ and $x = -2$ gives $10C = -1$, so $B = \frac{1}{5}$ and $C = -\frac{1}{10}$.

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Case II: $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we would use

$$\boxed{7} \qquad \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

Integration of Rational Functions by Partial Fractions

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

but we prefer to work out in detail a simpler example.

Example 4

Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx.$

Solution:

The first step is to divide. The result of long division is

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

Example 4 – *Solution*

cont'd

The second step is to factor the denominator $Q(x) = x^3 - x^2 - x + 1$.

Since $Q(1) = 0$, we know that $x - 1$ is a factor and we obtain

$$x^3 - x^2 - x + 1 = (x - 1)(x^2 - 1)$$

$$= (x - 1)(x - 1)(x + 1)$$

$$= (x - 1)^2(x + 1)$$

Example 4 – *Solution*

cont'd

Since the linear factor $x – 1$ occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiplying by the least common denominator, $(x – 1)^2(x + 1)$, we get

8 $$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Example 4 – *Solution*

cont'd

$$= (A + C)x^2 + (B - 2C)x + (-A + B + C)$$

Now we equate coefficients:

$$A + C = 0$$

$$B - 2C = 4$$

$$-A + B + C = 0$$

Example 4 – *Solution*

cont'd

Solving, we obtain $A = 1$, $B = 2$, and $C = -1$, so

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right] dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + K$$

Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \tag{9}$$

where A and B are constants to be determined.

Integration of Rational Functions by Partial Fractions

For instance, the function given by $f(x) = x/[(x - 2)(x^2 + 1)(x^2 + 4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

The term given in [9] can be integrated by completing the square (if necessary) and using the formula

10 $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Example 6

Evaluate $\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx.$

Solution:

Since the degree of the numerator is *not less than* the degree of the denominator, we first divide and obtain

$$\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} = 1 + \frac{x - 1}{4x^2 - 4x + 3}$$

Example 6 – *Solution*

cont'd

Notice that the quadratic $4x^2 - 4x + 3$ is irreducible because its discriminant is $b^2 - 4ac = -32 < 0$. This means it can't be factored, so we don't need to use the partial fraction technique.

To integrate the given function we complete the square in the denominator:

$$4x^2 - 4x + 3 = (2x - 1)^2 + 2$$

This suggests that we make the substitution $u = 2x - 1$.

Example 6 – *Solution*

cont'd

Then $du = 2\,dx$ and $x = \frac{1}{2}(u + 1)$, so

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx = \int \left(1 + \frac{x - 1}{4x^2 - 4x + 3}\right) dx$$

$$= x + \frac{1}{2}\int \frac{\frac{1}{2}(u + 1) - 1}{u^2 + 2}\,du$$

$$= x + \frac{1}{4}\int \frac{u - 1}{u^2 + 2}\,du$$

Example 6 – *Solution*

cont'd

$$= x + \tfrac{1}{4}\int \frac{u}{u^2 + 2}\,du - \tfrac{1}{4}\int \frac{1}{u^2 + 2}\,du$$

$$= x + \tfrac{1}{8}\ln(u^2 + 2) - \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= x + \tfrac{1}{8}\ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x - 1}{\sqrt{2}}\right) + C$$

Integration of Rational Functions by Partial Fractions

Note:
Example 6 illustrates the general procedure for integrating a partial fraction of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

where $b^2 - 4ac < 0$

We complete the square in the denominator and then make a substitution that brings the integral into the form

$$\int \frac{Cu + D}{u^2 + a^2}\,du = C\int \frac{u}{u^2 + a^2}\,du + D\int \frac{1}{u^2 + a^2}\,du$$

Then the first integral is a logarithm and the second is expressed in terms of $\tan^{-1}$.

Integration of Rational Functions by Partial Fractions

Case IV: $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction [9], the sum

$$\boxed{11} \qquad \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms in [11] can be integrated by using a substitution or by first completing the square if necessary.

Example 8

Evaluate $\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} dx.$

Solution:

The form of the partial fraction decomposition is

$$\frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

Multiplying by $x(x^2 + 1)^2$, we have

$$-x^3 + 2x^2 - x + 1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x$$

Example 8 – *Solution*

cont'd

$$= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex$$

$$= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A$$

If we equate coefficients, we get the system

$$A + B = 0 \qquad C = -1 \qquad 2A + B + D = 2 \qquad C + E = -1 \qquad A = 1$$

which has the solution $A = 1$, $B = -1$, $C = -1$, $D = 1$ and $E = 0$.

Example 8 – *Solution*

cont'd

Thus

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\,dx = \int \left(\frac{1}{x} - \frac{x + 1}{x^2 + 1} + \frac{x}{(x^2 + 1)^2} \right) dx$$

$$= \int \frac{dx}{x} - \int \frac{x}{x^2 + 1}\,dx - \int \frac{dx}{x^2 + 1} + \int \frac{x\,dx}{(x^2 + 1)^2}$$

$$= \ln|x| - \tfrac{1}{2}\ln(x^2 + 1) - \tan^{-1}x - \frac{1}{2(x^2 + 1)} + K$$

Rationalizing Substitutions

Rationalizing Substitutions

Some nonrational functions can be changed into rational functions by means of appropriate substitutions.

In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective. Other instances appear in the exercises.

Example 9

Evaluate $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx.$

Solution:

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u\,du$. Therefore

$$\int \frac{\sqrt{x+4}}{x}\,dx = \int \frac{u}{u^2-4}\,2u\,du$$

$$= 2\int \frac{u^2}{u^2-4}\,du$$

$$= 2\int \left(1 + \frac{4}{u^2-4}\right) du$$

Example 9 – *Solution*

cont'd

We can evaluate this integral either by factoring $u^2 - 4$ as $(u - 2)(u + 2)$ and using partial fractions or by using Formula 6 with $a = 2$:

$$\int \frac{\sqrt{x+4}}{x}\,dx = 2\int du + 8\int \frac{du}{u^2 - 4}$$

$$= 2u + 8 \cdot \frac{1}{2 \cdot 2} \ln \left| \frac{u-2}{u+2} \right| + C$$

$$= 2\sqrt{x+4} + 2\ln \left| \frac{\sqrt{x+4} - 2}{\sqrt{x+4} + 2} \right| + C$$