7 Techniques of Integration

7.5 Strategy for Integration

Strategy for Integration

In this section we present a collection of miscellaneous integrals in random order and the main challenge is to recognize which technique or formula to use.

No hard and fast rules can be given as to which method applies in a given situation, but we give some advice on strategy that you may find useful.

A prerequisite for applying a strategy is a knowledge of the basic integration formulas.

Strategy for Integration

In the table of Integration Formulas we have collected the integrals with several additional formulas that we have learned in this chapter.

Most of them should be memorized. It is useful to know them all, but the ones marked with an asterisk need not be memorized since they are easily derived.

Formula 19 can be avoided by using partial fractions, and trigonometric substitutions can be used in place of Formula 20.

Strategy for Integration

Table of Integration Formulas Constants of integration have been omitted.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\int \frac{1}{x}\,dx = \ln|x|$

3. $\int e^x\,dx = e^x$

4. $\int a^x\,dx = \frac{a^x}{\ln a}$

5. $\int \sin x\,dx = -\cos x$

6. $\int \cos x\,dx = \sin x$

7. $\int \sec^2 x\,dx = \tan x$

8. $\int \csc^2 x\,dx = -\cot x$

9. $\int \sec x \tan x\,dx = \sec x$

10. $\int \csc x \cot x\,dx = -\csc x$

11. $\int \sec x\,dx = \ln|\sec x + \tan x|$

12. $\int \csc x\,dx = \ln|\csc x - \cot x|$

13. $\int \tan x\,dx = \ln|\sec x|$

14. $\int \cot x\,dx = \ln|\sin x|$

15. $\int \sinh x\,dx = \cosh x$

16. $\int \cosh x\,dx = \sinh x$

17. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$

*19. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

*20. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Strategy for Integration

1. Simplify the Integrand if Possible

2. Sometimes the use of algebraic manipulation or trigonometric identities will simplify the integrand and make the method of integration obvious. Here are some examples:

$$\int \sqrt{x}\,(1 + \sqrt{x})\,dx = \int (\sqrt{x} + x)\,dx$$

$$\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cos^2\theta\,d\theta$$

3.

$$\int \sin\theta\cos\theta\,d\theta = \tfrac{1}{2}\int \sin 2\theta\,d\theta$$

Strategy for Integration

$$\int (\sin x + \cos x)^2\, dx = \int \sin^2 x + 2 \sin x \cos x + \cos^2 x)\, dx$$

$$= \int (1 + 2 \sin x \cos x)\, dx$$

2. Look for an Obvious Substitution

2. Try to find some function $u = g(x)$ in the integrand whose differential $du = g'(x)\, dx$ also occurs, apart from a constant factor. For instance, in the integral

$$\int \frac{x}{x^2 - 1}\, dx$$

we notice that if $u = x^2 - 1$, then $du = 2x\, dx$.

Strategy for Integration

Therefore we use the substitution $u = x^2 - 1$ instead of the method of partial fractions.

3. Classify the Integrand According to Its Form

If Steps 1 and 2 have not led to the solution, then we take a look at the form of the integrand $f(x)$.

(a) *Trigonometric functions*. If $f(x)$ is a product of powers of sin x and cos x, of tan x and sec x, or of cot x and csc x, then we use the substitutions.

(b) *Rational functions*. If f is a rational function, we use the procedure involving partial fractions.

Strategy for Integration

(c) *Integration by parts.* If *f*(*x*) is a product of a power of *x*

(or a polynomial) and a transcendental function (such as a trigonometric, exponential, or logarithmic function), then we try integration by parts, choosing *u* and *dv.*

(d) *Radicals.* Particular kinds of substitutions are reco $\sqrt{\pm x^2 \pm a^2}$ when certain radicals appear.

(i) If occurs, we use a trigonometric substitution.

$\sqrt[n]{ax + b}$

(ii) If oc $u = \sqrt[n]{ax + b}$. e the rationalizing substitution $\sqrt[n]{g(x)}$ ore generally, this sometimes works for .

Strategy for Integration

4. Try Again

If the first three steps have not produced the answer, remember that there are basically only two methods of integration: substitution and parts.

(a) *Try substitution.* Even if no substitution is obvious (Step 2), some inspiration or ingenuity (or even desperation) may suggest an appropriate substitution.

(b) *Try parts.* Although integration by parts is used most of the time on products of the form described in Step

Strategy for Integration

(c) *Manipulate the integrand*. Algebraic manipulations (perhaps rationalizing the denominator or using trigonometric identities) may be useful in transforming

the integral into an easier form. These manipulations may be more substantial than in Step 1 and may involve some ingenuity. Here is an example:

$$\int \frac{dx}{1 - \cos x} = \int \frac{1}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}\, dx = \int \frac{1 + \cos x}{1 - \cos^2 x}\, dx$$

$$= \int \frac{1 + \cos x}{\sin^2 x}\, dx = \int \left(\csc^2 x + \frac{\cos x}{\sin^2 x} \right) dx$$

Strategy for Integration

(d) *Relate the problem to previous problems*. When you have built up some experience in integration, you may be able to use a method on a given integral that is similar to a method you have already used on a previous integral. Or you may even be able to express the given integral in terms of a previous one.

Strategy for Integration

For instance, $\int \tan^2 x \sec x\, dx$ is a challenging integral,

but if we make use of the identity $\tan^2 x = \sec^2 x - 1$,we

can write

$$\int \tan^2 x \sec x\, dx = \int \sec^3 x\, dx - \int \sec x\, dx$$

and if $\int \sec^3 x\, dx$ has previously been evaluated, then that calculation can be used in the present problem.

Strategy for Integration

(e) *Use several methods*. Sometimes two or three methods are required to evaluate an integral. The evaluation could involve several successive substitutions of different types, or it might combine integration by parts with one or more substitutions.

Example 1

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx$$

In Step 1 we rewrite the integral:

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \tan^3 x \ \sec^3 x\,dx$$

The integral is now of the form $\int \tan^m x \sec^n x\,dx$ with m odd.

Alternatively, if in Step 1 we had written

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \frac{\sin^3 x}{\cos^3 x}\,\frac{1}{\cos^3 x}\,dx$$

Example 1

cont'd

$$= \int \frac{\sin^3 x}{\cos^6 x}\, dx$$

then we could have continued as follows with the substitution *u* = cos *x*:

$$\int \frac{\sin^3 x}{\cos^6 x}\, dx = \int \frac{1 - \cos^2 x}{\cos^6 x} \sin x\, dx$$

$$= \int \frac{1 - u^2}{u^6} (-du)$$

$$= \int \frac{u^2 - 1}{u^6}\, du$$

$$= \int (u^{-4} - u^{-6})\, du$$

Can We Integrate All Continuous Functions?

Can We Integrate All Continuous Functions?

The functions that we have been studied here are called **elementary functions**.

These are the polynomials, rational functions, power functions (x^a), exponential functions (a^x), logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition.

Can We Integrate All Continuous Functions?

For instance, the function

$$f(x) = \sqrt{\frac{x^2 - 1}{x^3 + 2x - 1}} + \ln(\cosh x) - xe^{\sin 2x}$$

is an elementary function.

If *f* is an elementary function, then *f′* is an elementary function but $\int f(x)\,dx$ need not be an elementary function.

Consider $f(x) = e^{x^2}$.

Can We Integrate All Continuous Functions?

Since *f* is continuous, its integral exists, and if we define the function *F* by

$$F(x) = \int_0^x e^{t^2}\,dt$$

then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

Thus *f*(*x*) = e^{x^2} has an antiderivative *F*, but it has been proved that *F* is not an elementary function.

This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}$ *dx* in terms of the functions we know.

Can We Integrate All Continuous Functions?

The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3 + 1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions don't have elementary antiderivatives.

You may be assured, though, that the integrals in the following exercises are all elementary functions.