9

Differential Equations

9.3 Separable Equations

Separable Equations

A **separable equation** is a first-order differential equation in which the expression for *dy*/*dx* can be factored as a function of *x* times a function of *y*.

In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name *separable* comes from the fact that the expression on the right side can be "separated" into a function of *x* and a function of *y*.

Separable Equations

Equivalently, if $f(y) \neq 0$, we could write

1 $$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = 1/f(y)$.

To solve this equation we rewrite it in the differential form

$$h(y)\,dy = g(x)\,dx$$

so that all y's are on one side of the equation and all x's are on the other side.

Separable Equations

Then we integrate both sides of the equation:

$$\boxed{2} \qquad \int h(y)\, dy = \int g(x)\, dx$$

Equation 2 defines *y* implicitly as a function of *x*. In some cases we may be able to solve for *y* in terms of *x*.

Separable Equations

We use the Chain Rule to justify this procedure: If *h* and *g* satisfy (2), then

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

so $\displaystyle \frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dx} = g(x)$

and

$$h(y)\,\frac{dy}{dx} = g(x)$$

Thus Equation 1 is satisfied.

Example 1

(a) Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Solution:

(a) We write the equation in terms of differentials and integrate both sides:

$$y^2\, dy = x^2\, dx$$

$$\int y^2 dy = \int x^2 dx$$

Example 1 – *Solution*

cont'd

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

where C is an arbitrary constant. (We could have used a constant C_1 on the left side and another constant C_2 on the right side. But then we could combine these constants by writing $C = C_2 - C_1$.)

Solving for y, we get

$$y = \sqrt[3]{x^3 + 3C}$$

Example 1 – *Solution*

cont'd

We could leave the solution like this or we could write it in the form

$$y = \sqrt[3]{x^3 + K}$$

where $K = 3C$. (Since C is an arbitrary constant, so is K.)

(b) If we put $x = 0$ in the general solution in part (a), we get

$y(0) = \sqrt[3]{K}$.

To $\sqrt[3]{K} = 2$ the initial condition $y(0) = 2$, we must have and so $K = 8$.

Thus the solution of the initial-value problem is

$$y = \sqrt[3]{x^3 + 8}$$

Orthogonal Trajectories

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles (see Figure 7).

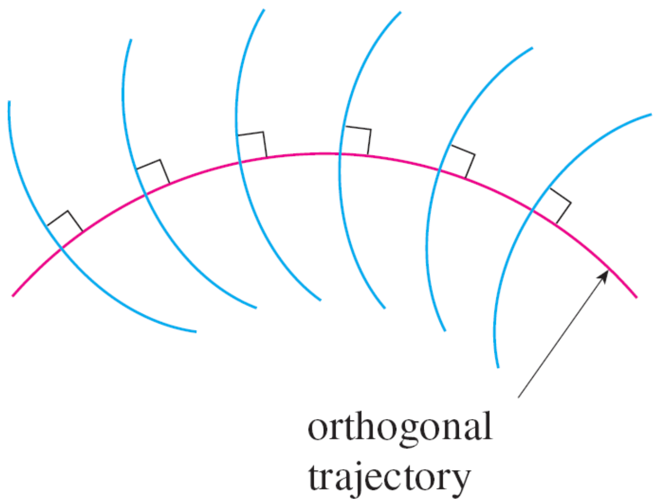

Figure 7

Orthogonal Trajectories

For instance, each member of the family $y = mx$ of straight lines through the origin is an orthogonal trajectory of the family $x^2 + y^2 = r^2$ of concentric circles with center the origin (see Figure 8). We say that the two families are orthogonal trajectories of each other.

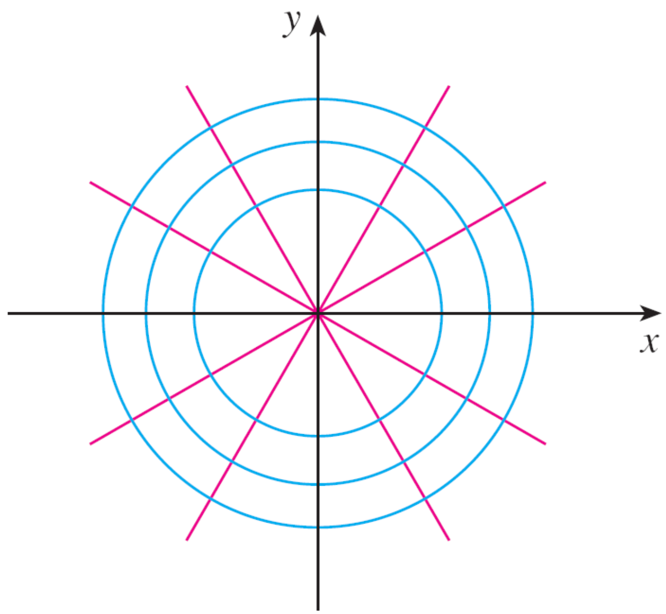

Figure 8

Example 5

Find the orthogonal trajectories of the family of curves $x = ky^2$, where is k an arbitrary constant.

Solution:

The curves $x = ky^2$ form a family of parabolas whose axis of symmetry is the x-axis.

The first step is to find a single differential equation that is satisfied by all members of the family.

If we differentiate $x = ky^2$, we get

$$1 = 2ky \frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{2ky}$$

Example 5 – *Solution*

cont'd

This differential equation depends on k, but we need an equation that is valid for all values of k simultaneously.

To eliminate k we note that, from the equation of the given general parabola $x = ky^2$, we have $k = x/y^2$ and so the differential equation can be written as

$$\frac{dy}{dx} = \frac{1}{2ky} = \frac{1}{2\frac{x}{y^2}y}$$

or

$$\frac{dy}{dx} = \frac{y}{2x}$$

Example 5 – *Solution*

cont'd

This means that the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = y/(2x)$.

On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope.

Therefore the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}$$

This differential equation is separable, and we solve it as follows:

$$\int y\,dy = -\int 2x\,dx$$

Example 5 – *Solution*

cont'd

$$\frac{y^2}{2} = -x^2 + C$$

4 $$x^2 + \frac{y^2}{2} = C$$

where *C* is an arbitrary positive constant.

Thus the orthogonal trajectories are the family of ellipses given by Equation 4 and sketched in Figure 9.

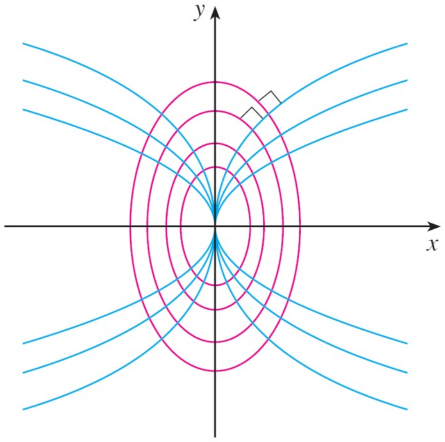

Figure 9

Mixing Problems

Mixing Problems

A typical mixing problem involves a tank of fixed capacity filled with a thoroughly mixed solution of some substance, such as salt.

A solution of a given concentration enters the tank at a fixed rate and the mixture, thoroughly stirred, leaves at a fixed rate, which may differ from the entering rate.

If $y(t)$ denotes the amount of substance in the tank at time t, then $y'(t)$ is the rate at which the substance is being added minus the rate at which it is being removed.

Mixing Problems

The mathematical description of this situation often leads to a first-order separable differential equation.

We can use the same type of reasoning to model a variety of phenomena: chemical reactions, discharge of pollutants into a lake, injection of a drug into the bloodstream.

Example 6

A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

Solution:

Let $y(t)$ be the amount of salt (in kilograms) after t minutes.

We are given that $y(0) = 20$ and we want to find $y(30)$. We do this by finding a differential equation satisfied by $y(t)$.

Example 6 – *Solution*

cont'd

Note that dy/dt is the rate of change of the amount of salt, so

$$\boxed{5} \qquad \frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

where (rate in) is the rate at which salt enters the tank and (rate out) is the rate at which salt leaves the tank.

We have

$$\text{rate in} = \left(0.03\,\frac{\text{kg}}{\text{L}}\right)\left(25\,\frac{\text{L}}{\text{min}}\right) = 0.75\,\frac{\text{kg}}{\text{min}}$$

The tank always contains 5000 L of liquid, so the concentration at time t is $y(t)/5000$ (measured in kilograms per liter).

Example 6 – *Solution*

cont'd

Since the brine flows out at a rate of 25 L/min, we have

$$\text{rate out} = \left(\frac{y(t)}{5000}\,\frac{\text{kg}}{\text{L}}\right)\left(25\,\frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\,\frac{\text{kg}}{\text{min}}$$

Thus, from Equation 5, we get

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

Solving this separable differential equation, we obtain

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$

$$-\ln|150 - y| = \frac{t}{200} + C$$

Example 6 – *Solution*

cont'd

Since $y(0) = 20$, we have $-\ln 130 = C$, so

$$-\ln|150 - y| = \frac{t}{200} - \ln 130$$

Therefore

$$|150 - y| = 130e^{-t/200}$$

Since $y(t)$ is continuous and $y(0) = 20$ and the right side is never 0, we deduce that $150 - y(t)$ is always positive.

Example 6 – *Solution*

cont'd

Thus $|150 - y| = 150 - y$ and so

$$y(t) = 150 - 130e^{-t/200}$$

The amount of salt after 30 min is

$$y(30) = 150 - 130e^{-30/200}$$

$$\approx 38.1\text{kg}$$