9

Differential Equations

9.4 Models for Population Growth

Models for Population Growth

In this section we investigate differential equations that are used to model population growth: the law of natural growth, the logistic equation, and several others.

The Law of Natural Growth

The Law of Natural Growth

In general, if $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then

$$\frac{dP}{dt} = kP \tag{1}$$

where k is a constant.

Equation 1 is sometimes called the **law of natural growth**. If k is positive, then the population increases; if k is negative, it decreases.

The Law of Natural Growth

Because Equation 1 is a separable differential equation, we can solve it by the methods given below:

$$\int \frac{dP}{P} = \int k\, dt$$

$$\ln |P| = kt + C$$

$$|P| = e^{kt + C} = e^{C}e^{kt}$$

$$P = Ae^{kt}$$

where A (= $\pm e^{C}$ or 0) is an arbitrary constant.

The Law of Natural Growth

To see the significance of the constant *A*, we observe that

$$P(0) = Ae^{k \cdot 0} = A$$

Therefore *A* is the initial value of the function.

2 The solution of the initial-value problem

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

The Law of Natural Growth

Another way of writing Equation 1 is

$$\frac{1}{P}\frac{dP}{dt} = k$$

which says that the **relative growth rate** (the growth rate divided by the population size) is constant.

Then $\boxed{2}$ says that a population with constant relative growth rate must grow exponentially.

The Law of Natural Growth

We can account for emigration (or "harvesting") from a population by modifying Equation 1: If the rate of emigration is a constant *m*, then the rate of change of the population is modeled by the differential equation

$$\boxed{3} \qquad \frac{dP}{dt} = kP - m$$

The Logistic Model

The Logistic Model

As we studied earlier, a population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources.

If $P(t)$ is the size of the population at time t, we assume that

$$\frac{dP}{dt} \approx kP \qquad \text{if } P \text{ is small}$$

This says that the growth rate is initially close to being proportional to size.

The Logistic Model

In other words, the relative growth rate is almost constant when the population is small. But we also want to reflect the fact that the relative growth rate decreases as the population P increases and becomes negative if P ever exceeds its **carrying capacity** M, the maximum population that the environment is capable of sustaining in the long run.

The simplest expression for the relative growth rate that incorporates these assumptions is

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

The Logistic Model

Multiplying by *P*, we obtain the model for population growth known as the **logistic differential equation**:

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$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Example 1

Draw a direction field for the logistic equation with $k = 0.08$ and carrying capacity $M = 1000$. What can you deduce about the solutions?

Solution:

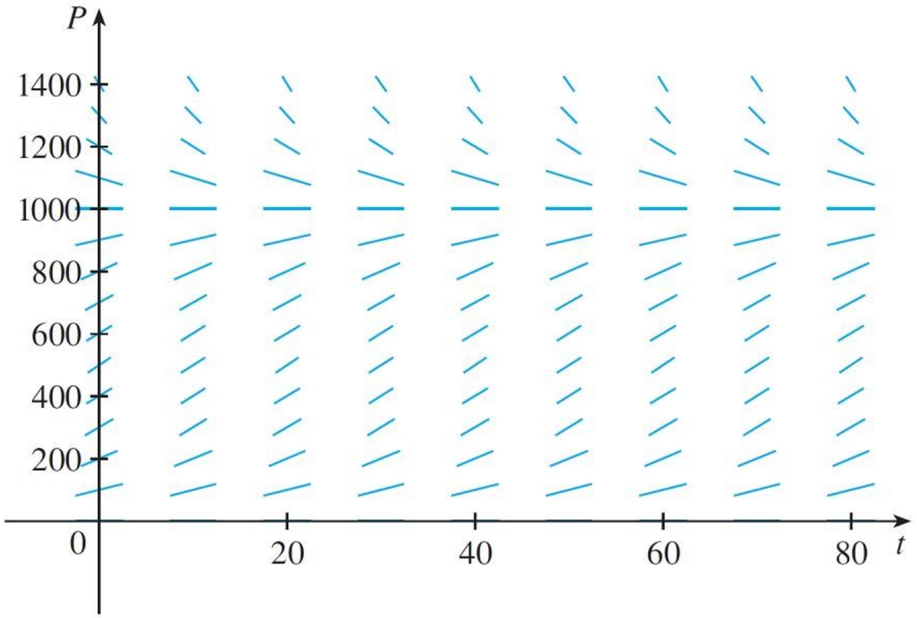
In this case the logistic differential equation is

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

Example 1 – *Solution*

cont'd

A direction field for this equation is shown in Figure 1.



Direction field for the logistic equation in Example 1

Figure 1

Example 1 – *Solution*

cont'd

We show only the first quadrant because negative populations aren't meaningful and we are interested only in what happens after $t = 0$.

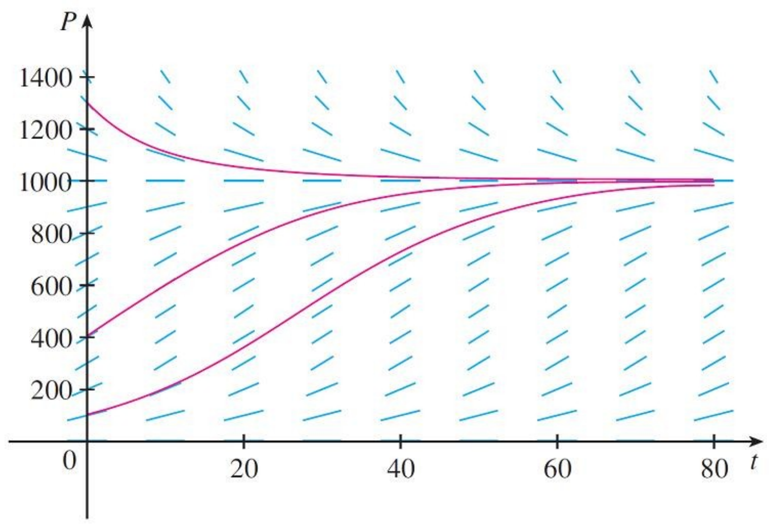
The logistic equation is autonomous (dP/dt depends only on P, not on t), so the slopes are the same along any horizontal line. As expected, the slopes are positive for $0 < P < 100$ and negative for $P > 1000$.

The slopes are small when P is close to 0 or 1000 (the carrying capacity). Notice that the solutions move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = 1000$.

Example 1 – *Solution*

cont'd

In Figure 2 we use the direction field to sketch solution curves with initial populations $P(0) = 100$, $P(0) = 400$, and $P(0) = 1300$.



Solution curves for the logistic equation in Example 1

Figure 2

Example 1 – *Solution*

cont'd

Notice that solution curves that start below $P = 1000$ are increasing and those that start above $P = 1000$ are decreasing.

The slopes are greatest when $P \approx 500$ and therefore the solution curves that start below $P = 1000$ have inflection points when $P \approx 500$.

In fact we can prove that all solution curves that start below $P = 500$ have an inflection point when P is exactly 500.

The Logistic Model

The logistic equation [4] is separable and so we can solve it explicitly. Since

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

we have

$$\boxed{5} \qquad \int \frac{dP}{P(1 - P/M)} = \int k\, dt$$

The Logistic Model

To evaluate the integral on the left side, we write

$$\frac{1}{P(1 - P/M)} = \frac{M}{P(M - P)}$$

Using partial fractions, we get

$$\frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}$$

This enables us to rewrite Equation 5:

$$\int \left(\frac{1}{P} + \frac{1}{M - P} \right) dP = \int k\, dt$$

The Logistic Model

$$\ln|P| - \ln|M - P| = kt + C$$

$$\ln\left|\frac{M - P}{P}\right| = -kt - C$$

$$\left|\frac{M - P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt}$$

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$$\frac{M - P}{P} = Ae^{-kt}$$

where $A = \pm e^{-C}$.

The Logistic Model

Solving Equation 6 for *P*, we get

$$\frac{M}{P} - 1 = Ae^{-kt} \qquad \Rightarrow \qquad \frac{P}{M} = \frac{1}{1 + Ae^{-kt}}$$

so

$$P = \frac{M}{1 + Ae^{-kt}}$$

The Logistic Model

We find the value of A by putting t = 0 in Equation 6. If t = 0, then P = P_0 (the initial population), so

$$\frac{M - P_0}{P_0} = Ae^0 = A$$

Thus the solution to the logistic equation is

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$$P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{M - P_0}{P_0}$$

The Logistic Model

Using the expression for $P(t)$ in Equation 7, we see that

$$\lim_{t \to \infty} P(t) = M$$

which is to be expected.

Example 2

Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

and use it to find the population sizes $P(40)$ and $P(80)$. At what time does the population reach 900?

Example 2 – *Solution*

The differential equation is a logistic equation with $k = 0.08$, carrying capacity $M = 1000$, and initial population $P_0 = 100$.

So Equation 7 gives the population at time t as

$$P(t) = \frac{1000}{1 + Ae^{-0.08t}}$$

where

$$A = \frac{1000 - 100}{100} = 9$$

Thus

$$P(t) = \frac{1000}{1 + 9e^{-0.08t}}$$

So the population sizes when $t = 40$ and 80 are

$$P(40) = \frac{1000}{1 + 9e^{-3.2}}$$

Example 2 – *Solution*

cont'd

$$\approx 731.6$$

$$P(80) = \frac{1000}{1 + 9e^{-6.4}}$$

$$\approx 985.3$$

The population reaches 900 when

$$\frac{1000}{1 + 9e^{-0.08t}} = 900$$

Example 2 – *Solution*

cont'd

Solving this equation for *t*, we get

$$1 + 9e^{-0.08t} = \frac{10}{9}$$

$$e^{-0.08t} = \frac{1}{81}$$

$$-0.08t = \ln \frac{1}{81}$$

$$= -\ln 81$$

$$t = \frac{\ln 81}{0.08}$$

$$\approx 54.9$$

So the population reaches 900 when *t* is approximately 55.

Example 2 – *Solution*

cont'd

As a check on our work, we graph the population curve in Figure 3 and observe where it intersects the line $P = 900$.

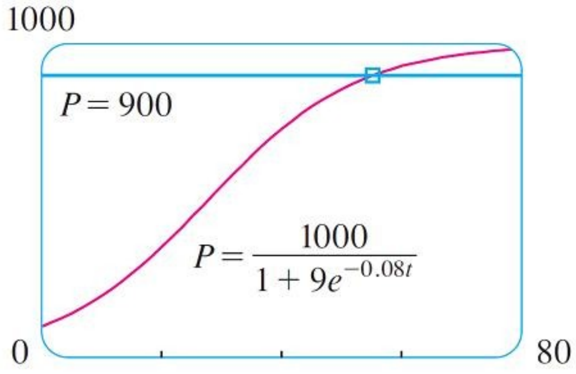


Figure 3

The cursor indicates that $t \approx 55$.

Comparison of the Natural Growth and Logistic Models

Comparison of the Natural Growth and Logistic Models

In the 1930s the biologist G. F. Gause conducted an experiment with the protozoan *Paramecium* and used a logistic equation to model his data. The table gives his daily count of the population of protozoa.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57

He estimated the initial relative growth rate to be 0.7944 and the carrying capacity to be 64.

Example 3

Find the exponential and logistic models for Gause's data. Compare the predicted values with the observed values and comment on the fit.

Solution:

Given the relative growth rate $k = 0.7944$ and the initial population $P_0 = 2$, the exponential model is

$$P(t) = P_0 e^{kt}$$

$$= 2e^{0.7944t}$$

Example 3 – *Solution*

cont'd

Gause used the same value of *k* for his logistic model. [This is reasonable because $P_0 = 2$ is small compared with the carrying capacity ($M = 64$).

The equation

$$\frac{1}{P_0}\frac{dP}{dt}\bigg|_{t=0} = k\left(1 - \frac{2}{64}\right) \approx k$$

shows that the value of *k* for the logistic model is very close to the value for the exponential model.]

7

$$P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{M - P_0}{P_0}$$

Example 3 – *Solution*

cont'd

Then the solution of the logistic equation in Equation 7 gives

$$P(t) = \frac{M}{1 + Ae^{-kt}}$$

$$= \frac{64}{1 + Ae^{-0.7944t}}$$

where

$$A = \frac{M - P_0}{P_0}$$

$$= \frac{64 - 2}{2} = 31$$

So

$$P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

Example 3 – *Solution*

cont'd

We use these equations to calculate the predicted values (rounded to the nearest integer) and compare them in the following table.

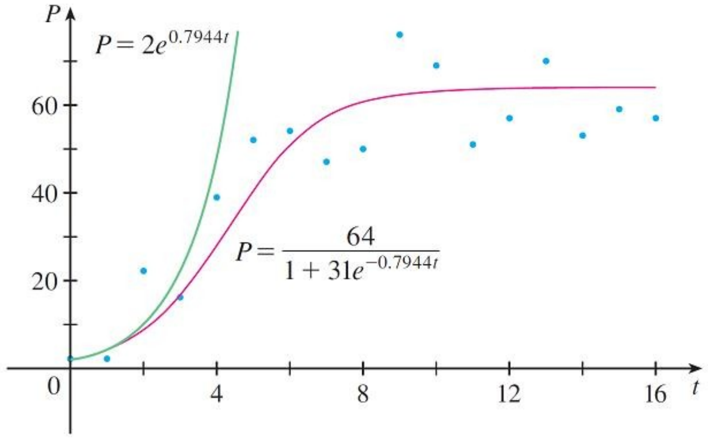
t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57
P (logistic model)	2	4	9	17	28	40	51	57	61	62	63	64	64	64	64	64	64
P (exponential model)	2	4	10	22	48	106	. . .										

Example 3 – *Solution*

cont'd

We notice from the table and from the graph in Figure 4 that for the first three or four days the exponential model gives results comparable to those of the more sophisticated logistic model.

For $t \geq 5$, however, the exponential model is hopelessly inaccurate, but the logistic model fits the observations reasonably well.



The exponential and logistic models for the *Paramecium* data

Figure 4

Other Models for Population Growth

Other Models for Population Growth

The Law of Natural Growth and the logistic differential equation are not the only equations that have been proposed to model population growth.

Two of the other models are modifications of the logistic model. The differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - c$$

has been used to model populations that are subject to harvesting of one sort or another. (Think of a population of fish being caught at a constant rate.)

Other Models for Population Growth

For some species there is a minimum population level *m* below which the species tends to become extinct. (Adults may not be able to find suitable mates.) Such populations have been modeled by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

where the extra factor, 1 – *m*/*p*, takes into account the consequences of a sparse population.