10

Parametric Equations and Polar Coordinates

10.2 Calculus with Parametric Curves

Tangents

Tangents

Suppose *f* and *g* are differentiable functions and we want to find the tangent line at a point on the curve where *y* is also a differentiable function of *x*.

Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Tangents

If $dx/dt \neq 0$, we can solve for dy/dx:

1

$$\boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0}$$

Equation 1 (which you can remember by thinking of canceling the *dt*’s) enables us to find the slope dy/dx of the tangent to a parametric curve without having to eliminate the parameter t.

Tangents

We see from $\boxed{1}$ that the curve has a horizontal tangent when $dy/dt = 0$ (provided that $dx/dt \neq 0$) and it has a vertical tangent when $dx/dt = 0$ (provided that $dy/dt \neq 0$).

This information is useful for sketching parametric curves.

It is also useful to consider d^2y/dx^2. This can be found by replacing y by dy/dx in Equation 1:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

Example 1

A curve *C* is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that *C* has two tangents at the point (3, 0) and find their equations.

(b) Find the points on *C* where the tangent is horizontal or vertical.

(c) Determine where the curve is concave upward or downward.

(d) Sketch the curve.

Example 1 – *Solution*

(a) Notice that $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$.

Therefore the point (3, 0) on C arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$.

This indicates that C crosses itself at (3, 0).

Example 1 – *Solution*

cont'd

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

$$= \frac{3}{2}\left(t - \frac{1}{t}\right)$$

the slope of the tangent when *t* = $\pm\sqrt{3}$ is *dy*/*dx* $= \pm 6/(2\sqrt{3}) = \pm\sqrt{3}$, so the equations of the tangents at (3, 0) are

$$y = \sqrt{3}\,(x - 3) \qquad \text{and} \qquad y = -\sqrt{3}\,(x - 3)$$

Example 1 – *Solution*

cont'd

(b) C has a horizontal tangent when $dy/dx = 0$, that is, when

$dy/dt = 0$ and $dx/dt \neq 0$. Since $dy/dt = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$.

The corresponding points on C are $(1, -2)$ and $(1, 2)$.

C has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$. (Note that $dy/dt \neq 0$ there.)

The corresponding point on C is $(0, 0)$.

Example 1 – *Solution*

cont’d

(c) To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{3}{2}\left(1 + \dfrac{1}{t^2}\right)}{2t}$$

$$= \frac{3(t^2 + 1)}{4t^3}$$

Thus the curve is concave upward when *t* > 0 and concave downward when *t* < 0.

Example 1 – *Solution*

cont'd

(d) Using the information from parts (b) and (c), we sketch *C* in Figure 1.

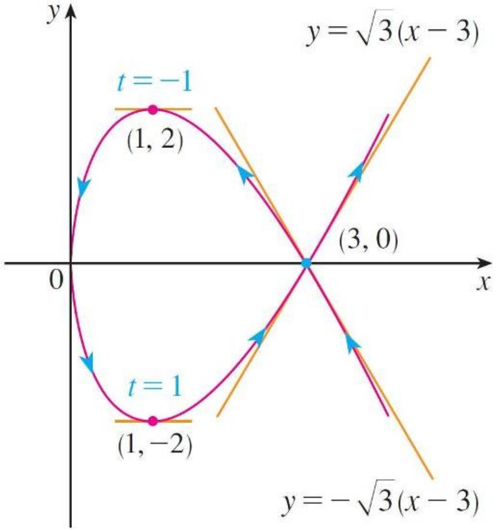


Figure 1

Areas

Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geq 0$.

If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt \qquad \left[\text{or} \quad \int_\beta^\alpha g(t) f'(t)\,dt\right]$$

Example 3

Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

(See Figure 3.)

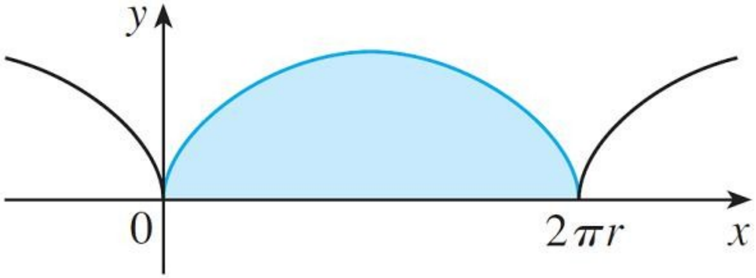


Figure 3

Example 3 – *Solution*

One arch of the cycloid is given by $0 \le \theta \le 2\pi$.

Using the Substitution Rule with $y = r(1 - \cos\theta)$ and $dx = r(1 - \cos\theta)\,d\theta$, we have

$$A = \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1 - \cos\theta)\,r(1 - \cos\theta)\,d\theta$$

$$= r^2 \int_0^{2\pi} (1 - \cos\theta)^2\,d\theta$$

$$= r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

Example 3 – *Solution*

cont'd

$$= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta$$

$$= r^2\left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= r^2\left(\tfrac{3}{2} \cdot 2\pi\right) = 3\pi r^2$$

Arc Length

Arc Length

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \leq x \leq b$.

If F' is continuous, then

$$\boxed{2} \qquad L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $dx/dt = f'(t) > 0$.

Arc Length

This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$.

Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

Since $dx/dt > 0$, we have

$$\boxed{3} \qquad L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Arc Length

Even if C can't be expressed in the form $y = F(x)$, Formula 3 is still valid but we obtain it by polygonal approximations.

We divide the parameter interval $[\alpha, \beta]$ into n subintervals of equal width Δt.

If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ approximates C.

Arc Length

(See Figure 4.)

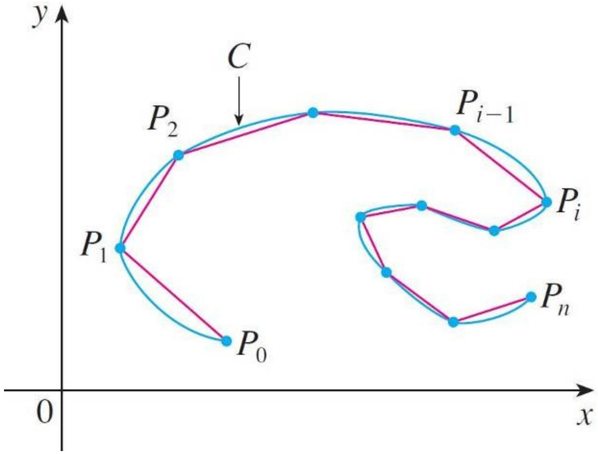


Figure 4

We define the length *L* of *C* to be the limit of the lengths of these approximating polygons as $n \to \infty$:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Arc Length

The Mean Value Theorem, when applied to *f* on the interval $[t_{i-1}, t_i]$, gives a number t_i^* n (t_{i-1}, t_i) such that

$$f(t_i) - f(t_{i-1}) = t_i^* \qquad)\,(t_i - t_{i-1})$$

If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, this equation becomes

$$t_i^* \qquad \Delta x_i = f'(\quad)\,\Delta t$$

Similarly, when applied to *g*, the Mean Value Theorem gives a number t_i^{**} in (t_{i-1}, t_i) such that

$$t_i^{**} \qquad \Delta y_i = g'(\quad)\,\Delta t$$

Arc Length

Therefore

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t_i^*)\,\Delta t]^2 + [g'(t_i^{**})\,\Delta t]^2}$$

$$= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\;\Delta t$$

and so

$$\boxed{4} \qquad L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\;\Delta t$$

Arc Length

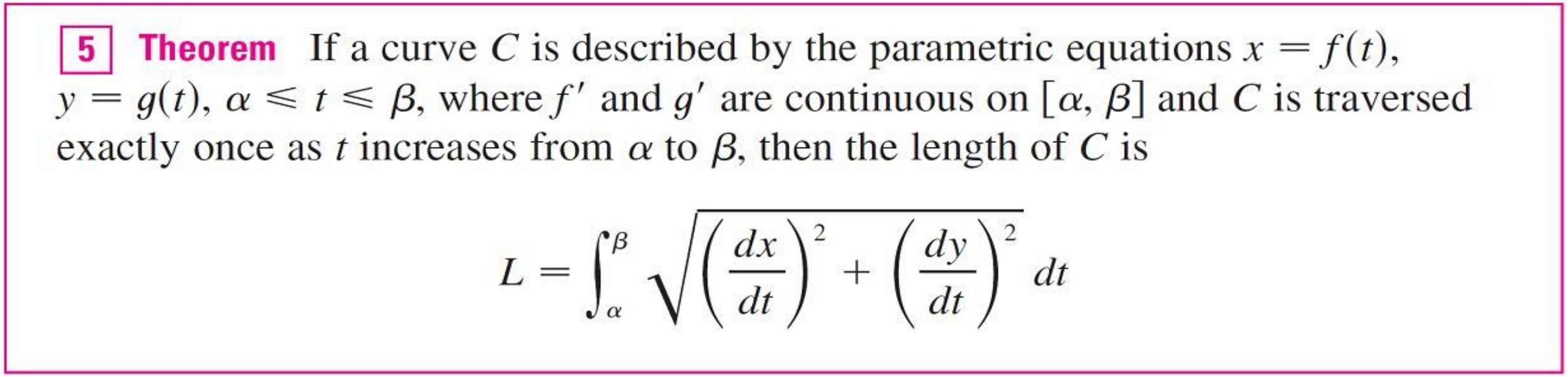
The sum in $\boxed{4}$ resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general.

Nevertheless, if *f′* and *g′* are continuous, it can be shown that the limit in $\boxed{4}$ is the same as if t_i^* and t_i^{**} were equal, namely,

$$L = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Arc Length

Thus, using Leibniz notation, we have the following result, which has the same form as Formula 3.



5 Theorem If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Notice that the formula in Theorem 5 is consistent with the general formulas $L = \int ds$ and $(ds)^2 = (dx)^2 + (dy)^2$.

Arc Length

Notice that the integral gives twice the arc length of the circle because as t increases from 0 to 2π, the point $(\sin 2t, \cos 2t)$ traverses the circle twice.

In general, when finding the length of a curve C from a parametric representation, we have to be careful to ensure that C is traversed only once as t increases from α to β.

Example 5

Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

Solution:

From Example 3 we see that one arch is described by the parameter interval $0 \le \theta \le 2\pi$.

Since

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \qquad \frac{dy}{d\theta} = r\sin\theta$$

Example 5 – *Solution*

cont'd

We have

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - \cos\theta)^2 + r^2 \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta$$

Example 5 – *Solution*

cont'd

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives $1 - \cos \theta = 2 \sin^2(\theta/2)$.

Since $0 \leq \theta \leq 2\pi$, we have $0 \leq \theta/2 \leq \pi$ and so $\sin(\theta/2) \geq 0$.

Therefore

$$\begin{aligned}\sqrt{2(1 - \cos\theta)} &= \sqrt{4 \sin^2(\theta/2)} \\ &= 2\left|\sin(\theta/2)\right| \\ &= 2 \sin(\theta/2)\end{aligned}$$

Example 5 – *Solution*

cont'd

so

$$L = 2r \int_0^{2\pi} \sin(\theta/2)\, d\theta$$

$$= 2r\Big[-2\cos(\theta/2)\Big]_0^{2\pi}$$

$$= 2r[2 + 2]$$

$$= 8r$$

Surface Area

Surface Area

If the curve given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, is rotated about the x-axis, where f', g' are continuous and $g(t) \geq 0$, then the area of the resulting surface is given by

$$\boxed{6} \qquad S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The general symbolic formulas $S = \int 2\pi y\, ds$ and $S = \int 2\pi x\, ds$ are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 6

Show that the surface area of a sphere of radius r is $4\pi r^2$.

Solution:

The sphere is obtained by rotating the semicircle

$$x = r\cos t \qquad y = r\sin t \qquad 0 \le t \le \pi$$

about the x-axis.

Therefore, from Formula 6, we get

$$S = \int_0^{\pi} 2\pi r \sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

Example 6 – *Solution*

cont'd

$$= 2\pi \int_0^\pi r \sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt$$

$$= 2\pi \int_0^\pi r \sin t \cdot r\, dt$$

$$= 2\pi r^2 \int_0^\pi \sin t\, dt$$

$$= 2\pi r^2(-\cos t)\Big]_0^\pi$$

$$= 4\pi r^2$$