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Parametric Equations and Polar Coordinates



10.4

Areas and Lengths in Polar Coordinates

Areas and Lengths in Polar Coordinates

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle:

$$1 \quad A = \frac{1}{2} r^2 \theta$$

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle.

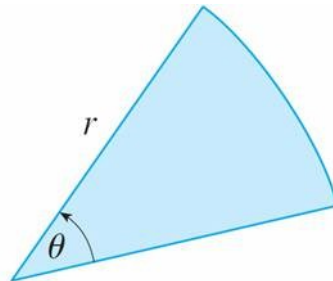


Figure 1

Areas and Lengths in Polar Coordinates

Formula 1 follows from the fact that the area of a sector is proportional to its central angle:

$$A = (\theta/2\pi)\pi r^2 = \frac{1}{2}r^2\theta.$$

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Let \mathcal{R} be the region, illustrated in Figure 2, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leq 2\pi$.

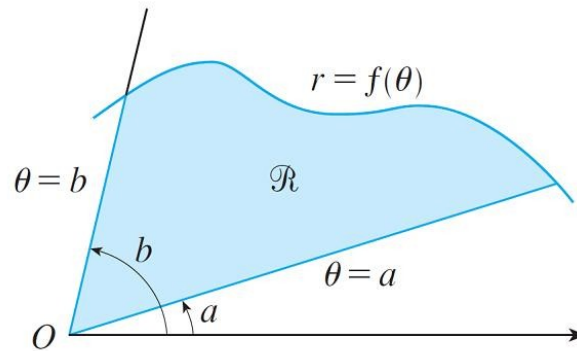


Figure 2

We divide the interval $[a, b]$ into subintervals with endpoints $\theta_0, \theta_1, \theta_2, \dots, \theta_n$ and equal width $\Delta\theta$.

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The rays $\theta = \theta_i$ then divide \mathcal{R} into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ_i^* in the i th subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the i th region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$. (See Figure 3.)

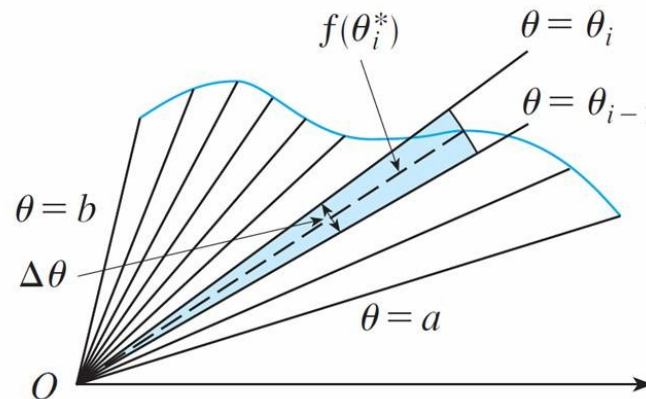


Figure 3

Areas and Lengths in Polar Coordinates

Thus from Formula 1 we have

$$\Delta A_i \approx \frac{1}{2} [f(\theta_i^*)]^2 \Delta \theta$$

and so an approximation to the total area A of \mathcal{R} is

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$$A \approx \sum_{i=1}^n \frac{1}{2} [f(\theta_i^*)]^2 \Delta \theta$$

It appears from Figure 3 that the approximation in 2 improves as $n \rightarrow \infty$.

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But the sums in $\boxed{2}$ are Riemann sums for the function $g(\theta) = \frac{1}{2} [f(\theta)]^2$, so

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$$

It therefore appears plausible that the formula for the area A of the polar region \mathcal{R} is

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$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$$

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Formula 3 is often written as

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$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

with the understanding that $r = f(\theta)$. Note the similarity between Formulas 1 and 4.

When we apply Formula 3 or 4 it is helpful to think of the area as being swept out by a rotating ray through O that starts with angle a and ends with angle b .

Example 1

Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution:

Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\pi/4$ to $\theta = \pi/4$.

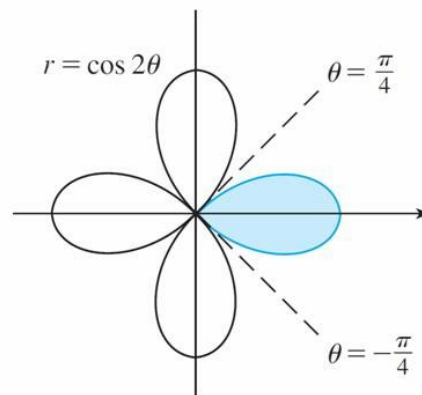


Figure 4

Example 1 – *Solution*

cont'd

Therefore Formula 4 gives

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta \\ &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta \\ &= \int_0^{\pi/4} \cos^2 2\theta d\theta \end{aligned}$$

Example 1 – *Solution*

cont'd

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2}(1 + \cos 4\theta) d\theta \\ &= \frac{1}{2} \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} \\ &= \frac{\pi}{8} \end{aligned}$$



Arc Length

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r \cos \theta = f(\theta) \cos \theta \qquad y = r \sin \theta = f(\theta) \sin \theta$$

Using the Product Rule and differentiating with respect to θ , we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

Arc Length

So, using $\cos^2 \theta + \sin^2 \theta = 1$, we have

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2 \theta - 2r \frac{dr}{d\theta} \cos \theta \sin \theta + r^2 \sin^2 \theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2 \theta + 2r \frac{dr}{d\theta} \sin \theta \cos \theta + r^2 \cos^2 \theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2\end{aligned}$$

Arc Length

Assuming that f' is continuous, we can write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

Therefore the length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is

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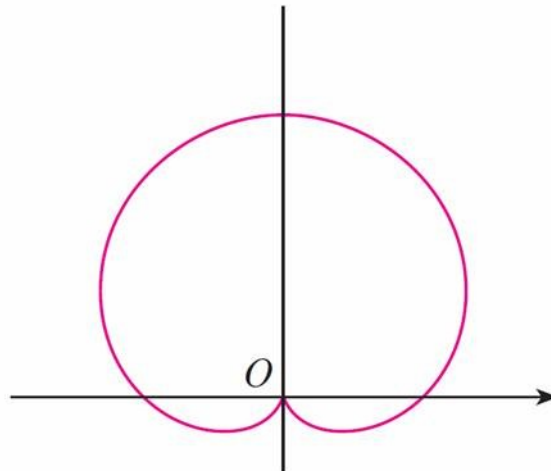
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Example 4

Find the length of the cardioid $r = 1 + \sin \theta$.

Solution:

The cardioid is shown in Figure 8.



$r = 1 + \sin$
 θ Figure 8

Example 4 – Solution

cont'd

5

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Its full length is given by the parameter interval $0 \leq \theta \leq 2\pi$, so Formula 5 gives

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{(1 + \sin \theta)^2 + \cos^2 \theta} d\theta \end{aligned}$$

Example 4 – *Solution*

cont'd

$$= \int_0^{2\pi} \sqrt{2 + 2 \sin \theta} \, d\theta$$

We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2 - 2 \sin \theta}$, or we could use a computer algebra system.

In any event, we find that the length of the cardioid is $L = 8$.