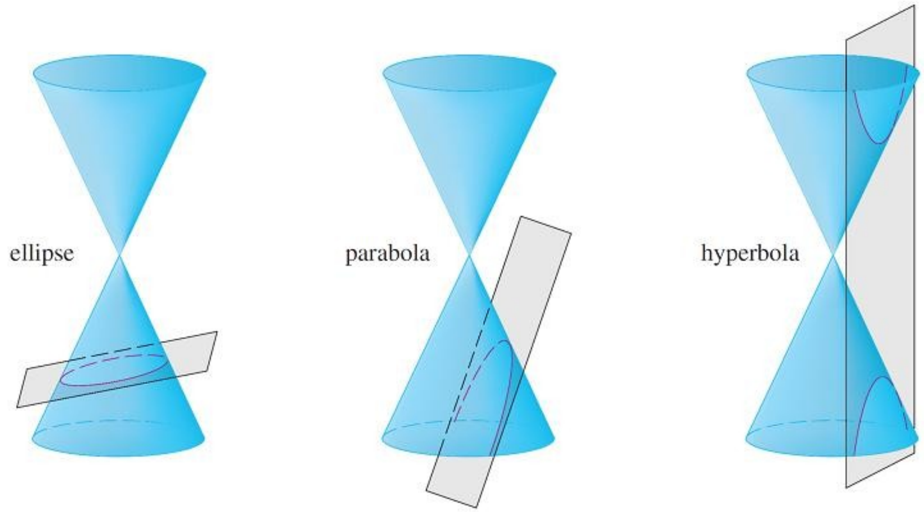
10

Parametric Equations and Polar Coordinates

10.5 Conic Sections

Conic Sections

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 1.



Conics

Figure 1

Parabolas

Parabolas

A **parabola** is the set of points in a plane that are equidistant from a fixed point *F* (called the **focus**) and a fixed line (called the **directrix**). This definition is illustrated by Figure 2.

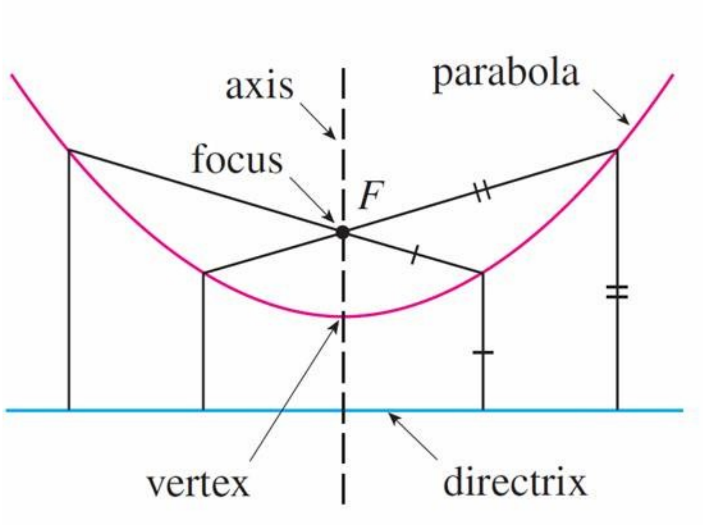


Figure 2

Parabolas

Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**.

The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges.

Parabolas

We obtain a particularly simple equation for a parabola if we place its vertex at the origin *O* and its directrix parallel to the *x*-axis as in Figure 3.

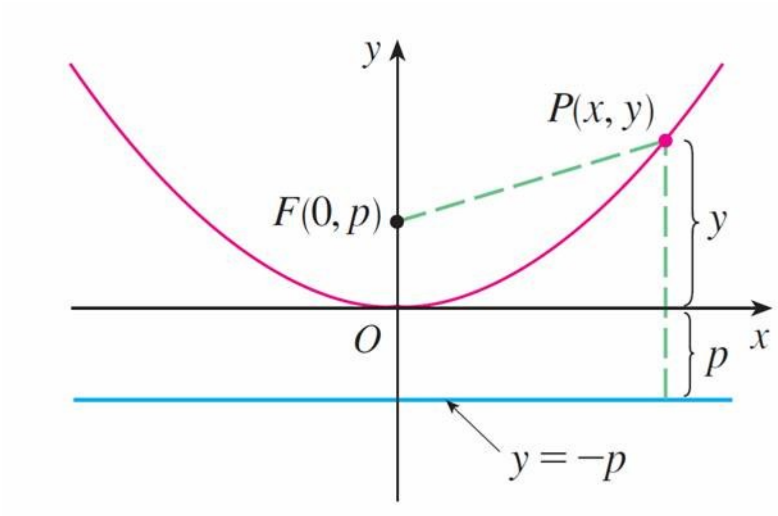


Figure 3

Parabolas

If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$. If $P(x, y)$ is any point on the parabola, then the distance from P to the focus is

$$|PF| = \sqrt{x^2 + (y - p)^2}$$

and the distance from P to the directrix is $|y + p|$. (Figure 3 illustrates the case where $p > 0$.)

Parabolas

The defining property of a parabola is that these distances are equal:

$$\sqrt{x^2 + (y - p)^2} = |y + p|$$

We get an equivalent equation by squaring and simplifying:

$$x^2 + (y - p)^2 = |y + p|^2 = (y + p)^2$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

Parabolas

$$x^2 = 4py$$

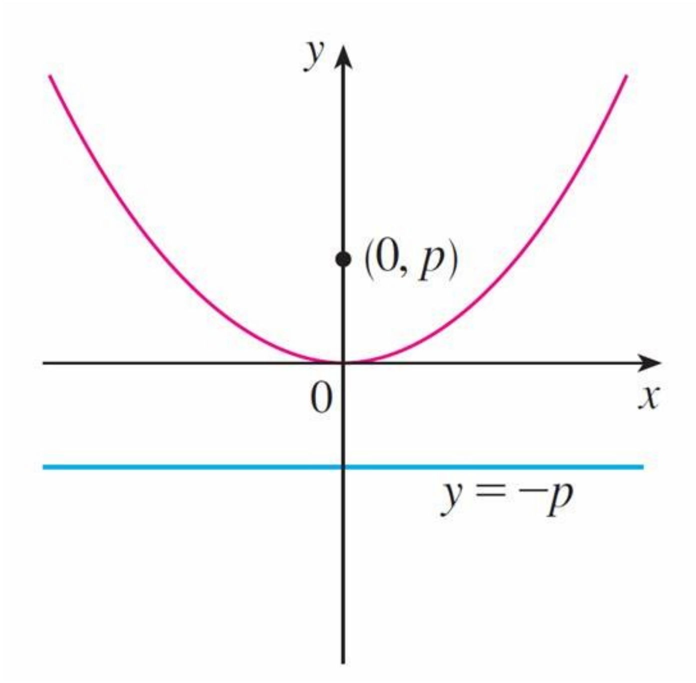
1 An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

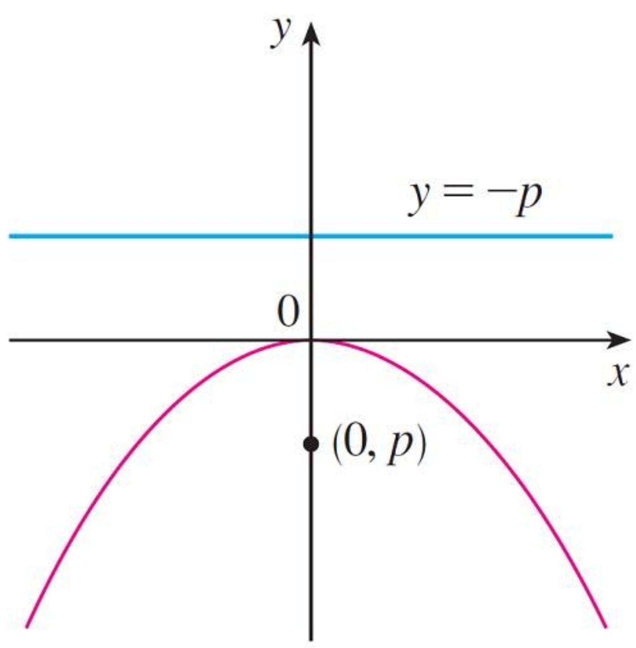
If we write $a = 1/(4p)$, then the standard equation of a parabola 1 becomes $y = ax^2$.

Parabolas

It opens upward if $p > 0$ and downward if $p < 0$ [see Figure 4, parts (a) and (b)].



(a) $x^2 = 4py$, $p > 0$



(b) $x^2 = 4py$, $p < 0$

Figure 4

Parabolas

The graph is symmetric with respect to the *y*-axis because [1] is unchanged when is replaced by –*x*.

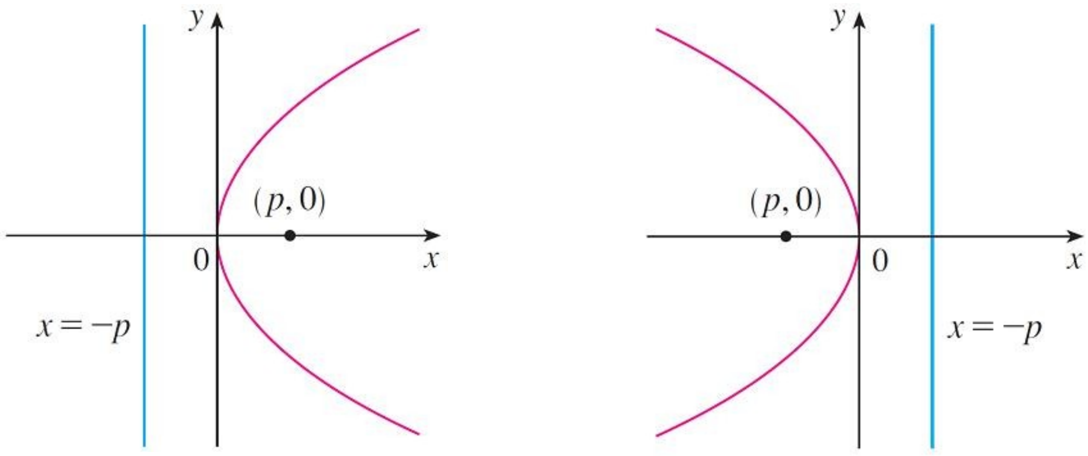
If we interchange *x* and *y* in [1], we obtain

2 $$y^2 = 4px$$

which is an equation of the parabola with focus (p, 0) and directrix $x = -p$. (Interchanging *x* and *y* amounts to reflecting about the diagonal line $y = x$.)

Parabolas

The parabola opens to the right if $p > 0$ and to the left if $p < 0$ [see Figure 4, parts (c) and (d)].



(c) $y^2 = 4px$, $p > 0$

(d) $y^2 = 4px$, $p < 0$

Figure 4

In both cases the graph is symmetric with respect to the *x*-axis, which is the axis of the parabola.

Example 1

Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

Solution:

If we write the equation as $y^2 = -10x$ and compare it with Equation 2, we see that $4p = -10$, so $p = -\frac{5}{2}$.

Thus the focus is $(p, 0) = (-\frac{5}{2}, 0)$ and the directrix is $x = \frac{5}{2}$.

Example 1 – *Solution*

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The sketch is shown in Figure 5.

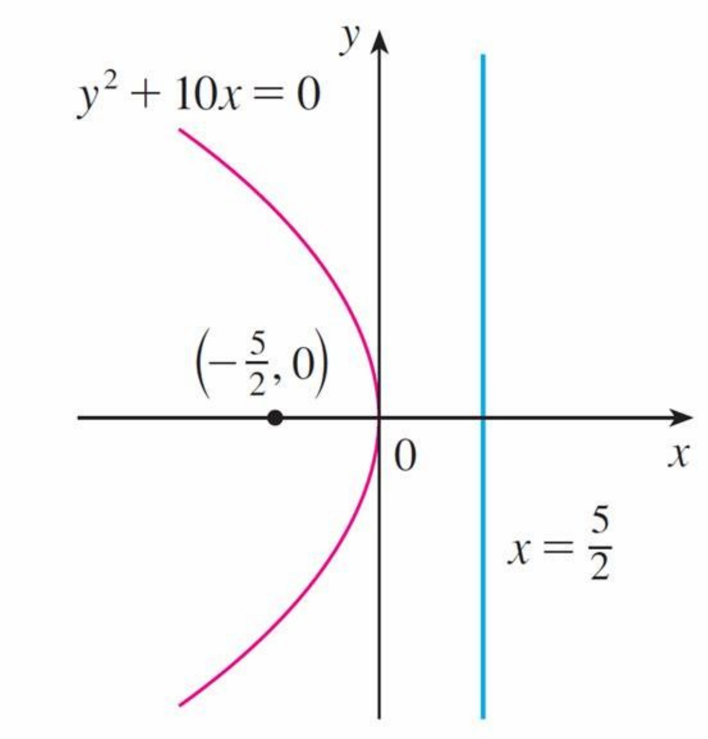


Figure 5

Ellipses

Ellipses

An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant (see Figure 6).

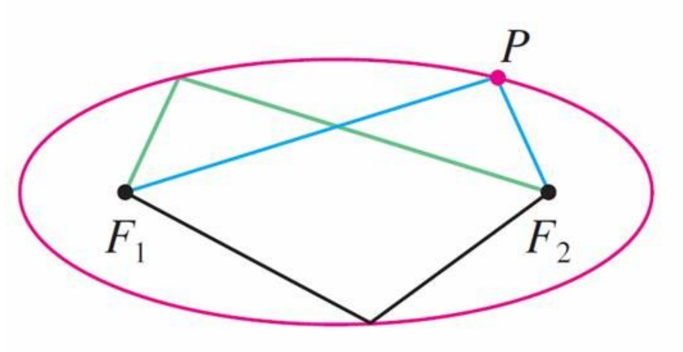


Figure 6

These two fixed points are called the **foci** (plural of **focus**). One of Kepler's laws is that the orbits of the planets in the solar system are ellipses with the sun at one focus.

Ellipses

In order to obtain the simplest equation for an ellipse, we place the foci on the *x*-axis at the points (– *c*, 0) and (*c*, 0) as in Figure 7 so that the origin is halfway between the foci.

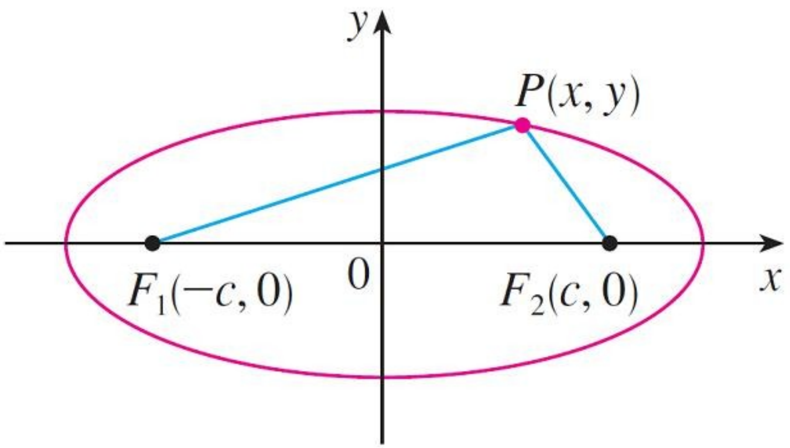


Figure 7

Ellipses

Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Then $P(x, y)$ is a point on the ellipse when

$$|PF_1| + |PF_2| = 2a$$

that is,

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

or

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

Ellipses

Squaring both sides, we have

$$x^2 - 2cx + c^2 + y^2$$

$$= 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

which simplifies to

$$a\sqrt{(x + c)^2 + y^2} = a^2 + cx$$

We square again:

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2$$

which becomes

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Ellipses

From triangle F_1F_2P in Figure 7 we see that $2c < 2a$, so $c < a$ and therefore $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$.

Then the equation of the ellipse becomes $b^2x^2 + a^2y^2 = a^2b^2$ or, if both sides are divided by a^2b^2,

3
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since $b^2 = a^2 - c^2 < a^2$, it follows that $b < a$.

Ellipses

The x-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or $x^2 = a^2$, so $x = \pm a$.

The corresponding points (a, 0) and ($-a$, 0) are called the **vertices** of the ellipse and the line segment joining the vertices is called the **major axis**. To find the y-intercepts we set $x = 0$ and obtain $y^2 = b^2$, so $y = \pm b$.

The line segment joining (0, b) and (0, $-b$) is the **minor axis**.

Ellipses

Equation 3 is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric about both axes.

Notice that if the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

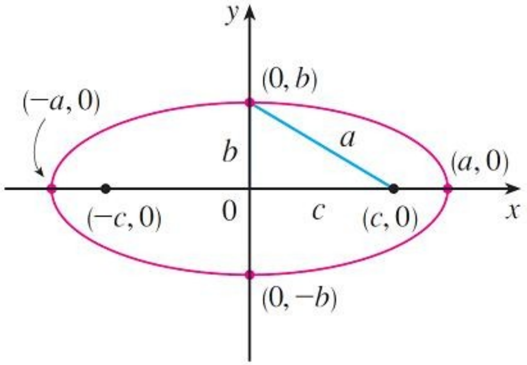
Ellipses

We summarize this discussion as follows (see also Figure 8).

4 The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geqslant b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

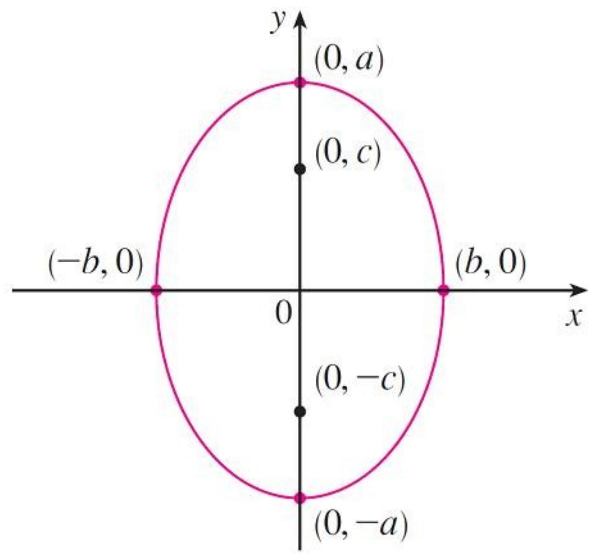


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ a \geqslant b$$

Figure 8

Ellipses

If the foci of an ellipse are located on the *y*-axis at $(0, \pm c)$, then we can find its equation by interchanging *x* and *y* in [4]. (See Figure 9.)



$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad a \geqslant b$$

Figure 9

Ellipses

5 The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geqslant b > 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

Example 2

Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

Solution:

Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

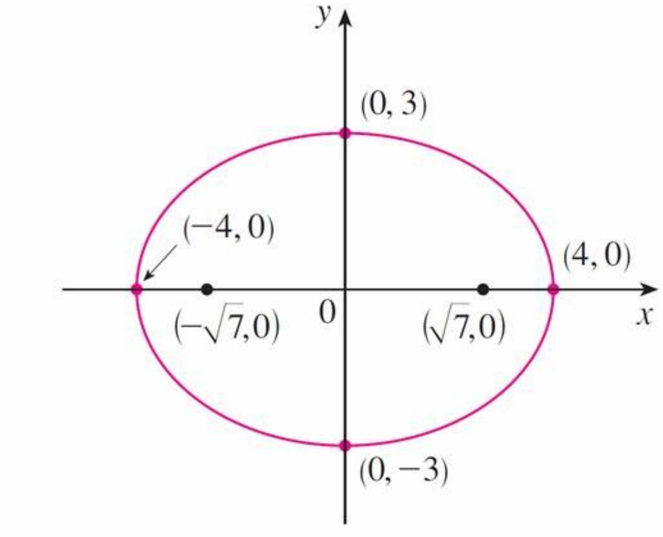
The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$.

The x-intercepts are ± 4 and the y-intercepts are ± 3.

Example 2 – *Solution*

cont'd

Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$ and the foci are $(\pm\sqrt{7}, 0)$. The graph is sketched in Figure 10.



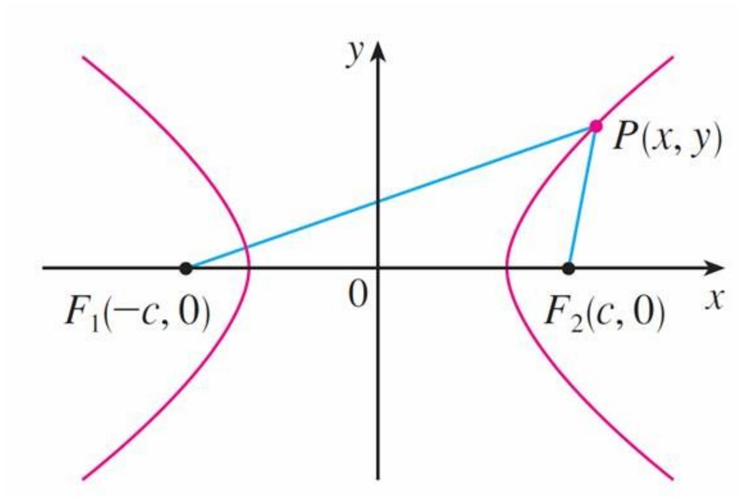
$9x^2 + 16y^2 = 144$

Figure 10

Hyperbolas

Hyperbolas

A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant. This definition is illustrated in Figure 11.



P is on the hyperbola when

$|PF_1| - |PF_2| = \pm 2a$.

Figure 11

Hyperbolas

Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances.

In fact, the derivation of the equation of a hyperbola is also similar to the one given earlier for an ellipse.

When the foci are on the x-axis at $(\pm c, 0)$ and the difference of distances is $|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is

6 $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$.

Hyperbolas

Notice that the x-intercepts are again $\pm a$ and the points $(a, 0)$ and $(-a, 0)$ are the **vertices** of the hyperbola.

But if we put $x = 0$ in Equation 6 we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation 6 and obtain

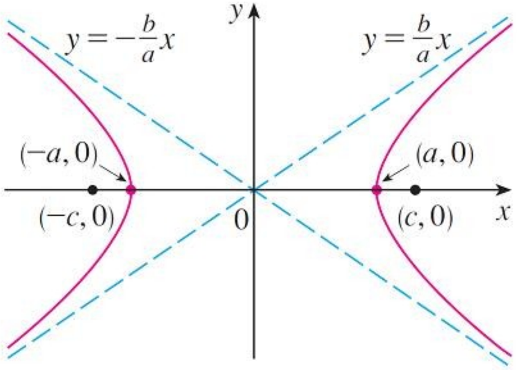
$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

This shows that $x^2 \geq a^2$, so $|x| = \sqrt{x^2} \geqslant a$.

Hyperbolas

Therefore we have $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its *branches.*

When we draw a hyperbola it is useful to first draw its **asymptotes**, which are the dashed lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 12.



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Figure 12

Hyperbolas

Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes.

7 The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

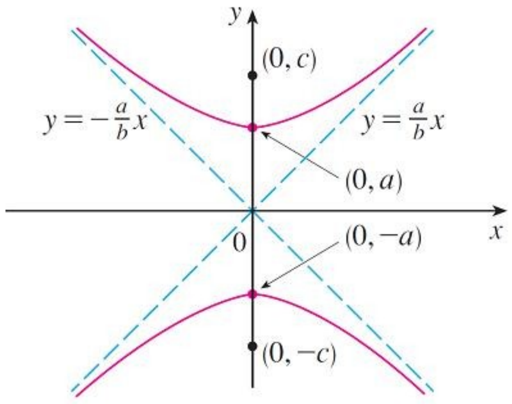
Hyperbolas

If the foci of a hyperbola are on the *y*-axis, then by reversing the roles of *x* and *y* we obtain the following information, which is illustrated in Figure 13.

8 The hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$.



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Figure 13

Example 4

Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

Solution:

If we divide both sides of the equation by 144, it becomes

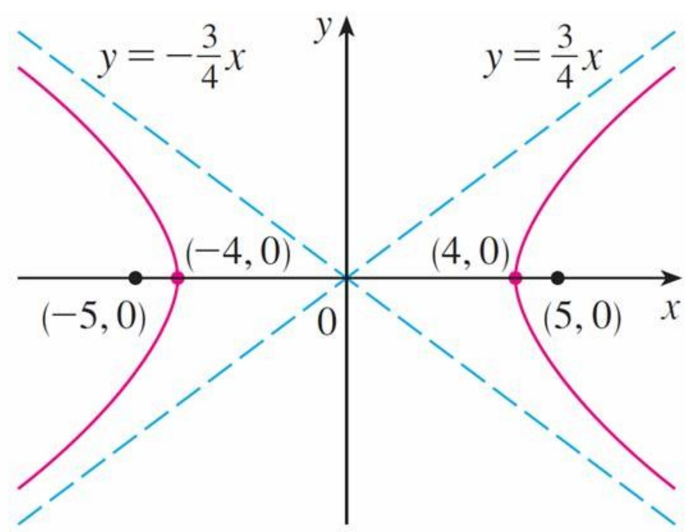
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is of the form given in $\boxed{7}$ with $a = 4$ and $b = 3$.

Example 4 – *Solution*

cont'd

Since $c^2 = 16 + 9 = 25$, the foci are $(\pm 5, 0)$. The asymptotes are the lines $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$. The graph is shown in Figure 14.



$9x^2 - 16y^2 = 144$

Figure 14

Shifted Conics

Shifted Conics

We shift conics by taking the standard equations [1], [2], [4],
[5], [7],   [8]
and   and replacing *x* and *y* by *x* – *h* and *y* – *k*.

Example 6

Find an equation of the ellipse with foci (2, –2), (4, –2) and vertices (1, –2), (5, –2).

Solution:

The major axis is the line segment that joins the vertices (1, –2), (5, –2) and has length 4, so $a = 2$. The distance between the foci is 2, so $c = 1$. Thus $b^2 = a^2 - c^2 = 3$. Since the center of the ellipse is (3, –2), we replace x and y in $\boxed{4}$ by $x - 3$ and $y + 2$ to obtain

$$\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1$$

as the equation of the ellipse.

Example 7

Sketch the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find its foci.

Solution:

We complete the squares as follows:

$$4(y^2 - 2y) - 9(x^2 - 8x) = 176$$

$$4(y^2 - 2y + 1) - 9(x^2 - 8x + 16) = 176 + 4 - 144$$

$$4(y - 1)^2 - 9(x - 4)^2 = 36$$

Example 7 – *Solution*

cont'd

$$\frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$$

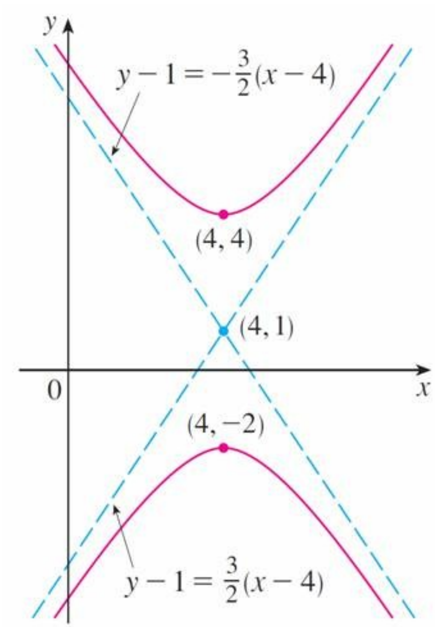
This is in the form $\boxed{8}$ except that x and y are replaced by $x - 4$ and $y - 1$. Thus $a^2 = 9$, $b^2 = 4$, and $c^2 = 13$.

The hyperbola is shifted four units to the right and one unit upward.

Example 7 – *Solution*

cont'd

The foci are $(4, 1 + \sqrt{13})$ and $(4, 1 - \sqrt{13})$ and the vertices are (4, 4) and (4, –2). The asymptotes are $y - 1 = \pm\frac{3}{2}(x - 4)$. The hyperbola is sketched in Figure 15.



$9x^2 - 4y^2 - 72x + 8y + 176 = 0$

Figure 15