

11

Infinite Sequences and Series



11.1

Sequences

Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1} .

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers.

Sequences

But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n .

Notation: The sequence $\{a_1, a_2, a_3, \dots\}$ is also denoted by

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

Example 1

Some sequences can be defined by giving a formula for the n th term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that n doesn't have to start at 1.

$$(a) \quad \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \quad a_n = \frac{n}{n+1} \quad \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots \right\}$$

$$(b) \quad \left\{ \frac{(-1)^n(n+1)}{3^n} \right\} \quad a_n = \frac{(-1)^n(n+1)}{3^n}$$

$$\left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots \right\}$$

Example 1

cont'd

$$(c) \quad \left\{ \sqrt{n-3} \right\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3}, \quad n \geq 3$$

$$\{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\}$$

$$(d) \quad \left\{ \cos \frac{n\pi}{6} \right\}_{n=0}^{\infty} \quad a_n = \cos \frac{n\pi}{6}, \quad n \geq 0$$

$$\left\{ 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots, \cos \frac{n\pi}{6}, \dots \right\}$$

Sequences

A sequence such as the one in Example 1(a), $a_n = n/(n + 1)$, can be pictured either by plotting its terms on a number line, as in Figure 1, or by plotting its graph, as in Figure 2.

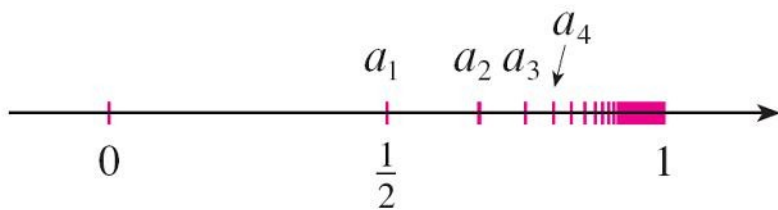


Figure 1

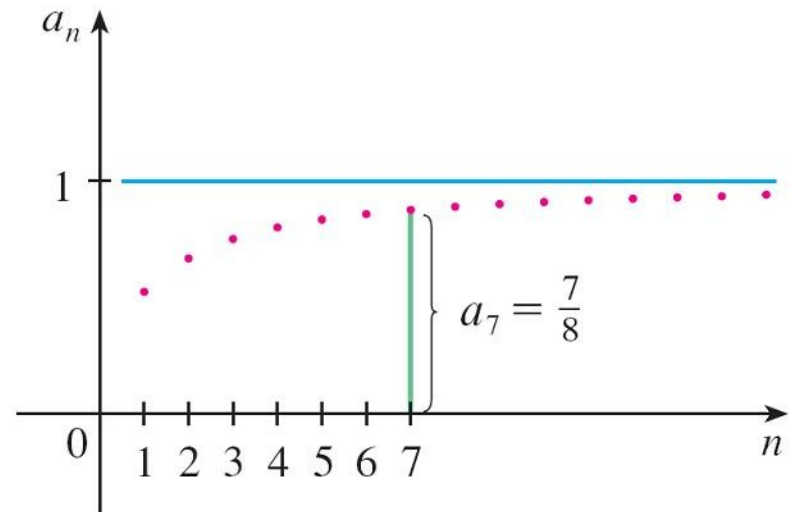


Figure 2

Sequences

Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1) \quad (2, a_2) \quad (3, a_3) \quad \dots \quad (n, a_n) \quad \dots$$

From Figure 1 or Figure 2 it appears that the terms of the sequence $a_n = n/(n + 1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n + 1} = \frac{1}{n + 1}$$

can be made as small as we like by taking n sufficiently large.

Sequences

We indicate this by writing

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n \rightarrow \infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large.

Sequences

Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity.

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \rightarrow \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Sequences

Figure 3 illustrates Definition 1 by showing the graphs of two sequences that have the limit L .

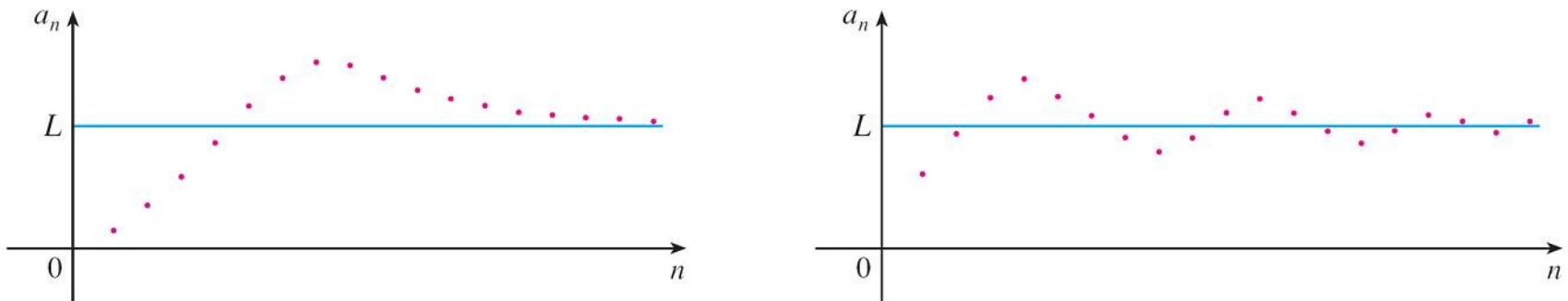


Figure 3

Graphs of two sequences with $\lim_{n \rightarrow \infty} a_n = L$

Sequences

A more precise version of Definition 1 is as follows.

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if } n > N \quad \text{then} \quad |a_n - L| < \varepsilon$$

Sequences

Definition 2 is illustrated by Figure 4, in which the terms a_1, a_2, a_3, \dots are plotted on a number line.

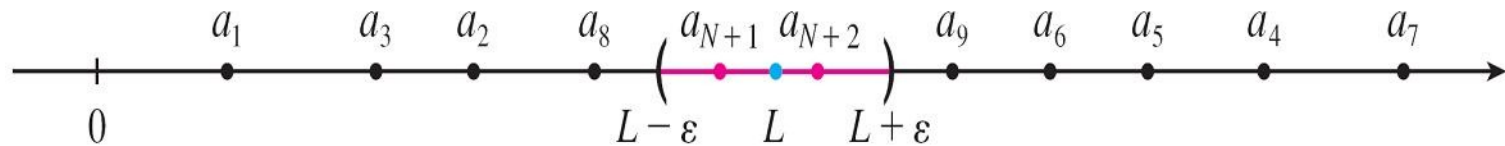


Figure 4

No matter how small an interval $(L - \epsilon, L + \epsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

Sequences

Another illustration of Definition 2 is given in Figure 5. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N .

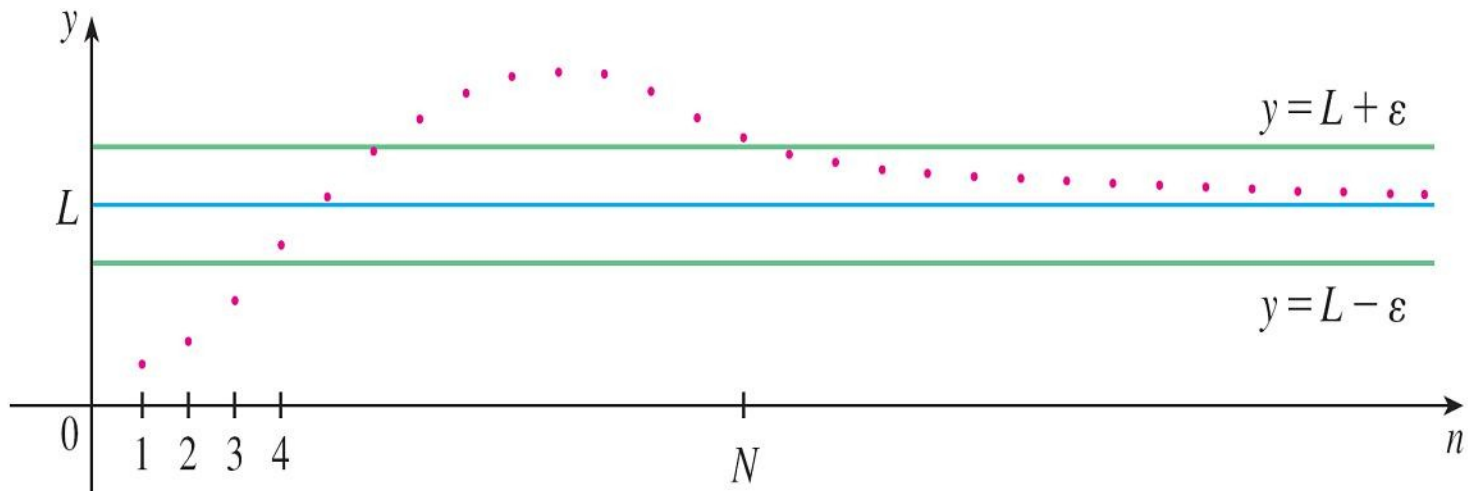


Figure 5

Sequences

You will see that the only difference between $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{x \rightarrow \infty} f(x) = L$ is that n is required to be an integer. Thus we have the following theorem, which is illustrated by Figure 6.

3 Theorem If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n \rightarrow \infty} a_n = L$.

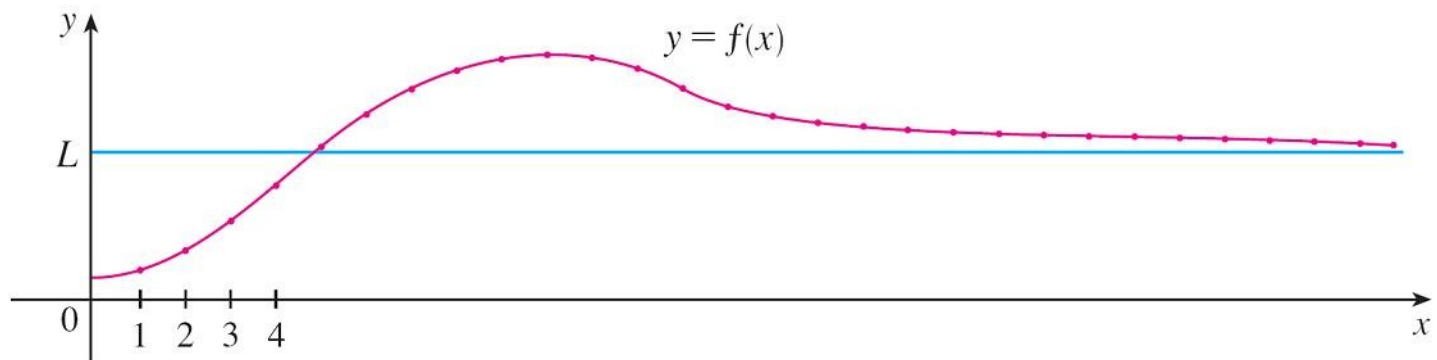


Figure 6

Sequences

In particular, since we know that $\lim_{x \rightarrow \infty} (1/x^r) = 0$ when $r > 0$, we have

$$\boxed{4} \quad \lim_{n \rightarrow \infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n \rightarrow \infty} a_n = \infty$. Consider the definition

5 Definition $\lim_{n \rightarrow \infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if } n > N \quad \text{then } a_n > M$$

If $\lim_{n \rightarrow \infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞ .

Sequences

The Limit Laws also hold for the limits of sequences and their proofs are similar.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n$$

$$\lim_{n \rightarrow \infty} c = c$$

$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \quad \text{if } \lim_{n \rightarrow \infty} b_n \neq 0$$

$$\lim_{n \rightarrow \infty} a_n^p = \left[\lim_{n \rightarrow \infty} a_n \right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Sequences

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 7).

Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.

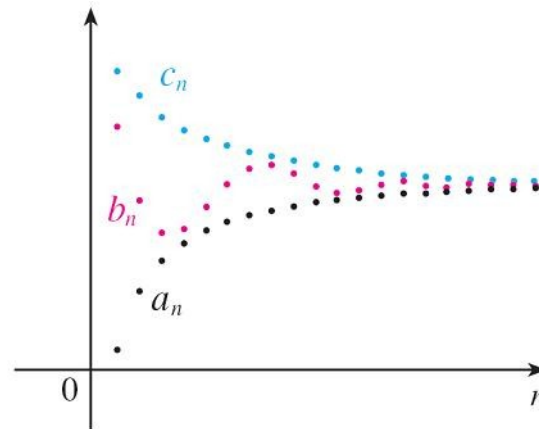


Figure 7

The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

Sequences

Another useful fact about limits of sequences is given by the following theorem.

6 Theorem

If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent.

7 Theorem

If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L , then

$$\lim_{n \rightarrow \infty} f(a_n) = f(L)$$

Example 11

For what values of r is the sequence $\{r^n\}$ convergent?

Solution:

We know that $\lim_{x \rightarrow \infty} a^x = \infty$ for $a > 1$ and $\lim_{x \rightarrow \infty} a^x = 0$ for $0 < a < 1$. Therefore, putting $a = r$ and using Theorem 3, we have

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

It is obvious that

$$\lim_{n \rightarrow \infty} 1^n = 1$$

$$\lim_{n \rightarrow \infty} 0^n = 0$$

Example 11 – *Solution*

cont'd

If $-1 < r < 0$, then $0 < |r| < 1$, so

$$\lim_{n \rightarrow \infty} |r^n| = \lim_{n \rightarrow \infty} |r|^n = 0$$

and therefore $\lim_{n \rightarrow \infty} r^n = 0$ by Theorem 6.

If $r \leq -1$, then $\{r^n\}$ diverges.

Example 11 – Solution

cont'd

Figure 11 shows the graphs for various values of r . (The case $r = -1$ is shown in Figure 8.)

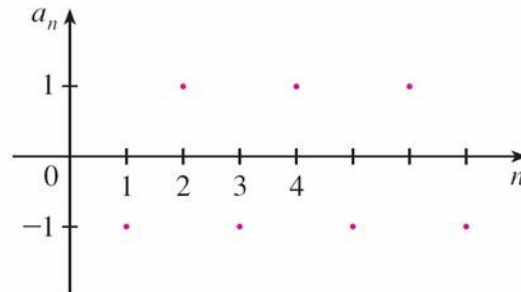


Figure 8

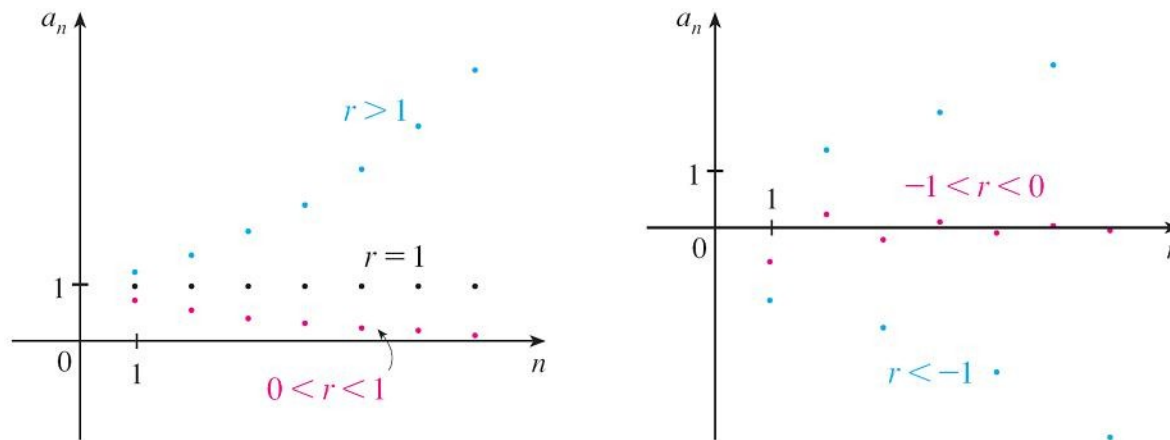


Figure 11

The sequence $a_n = r^n$

Sequences

The results of Example 11 are summarized as follows.

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \dots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Sequences

11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{for all } n \geq 1$$

It is **bounded below** if there is a number m such that

$$m \leq a_n \quad \text{for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below ($a_n > 0$) but not above. The sequence $a_n = n/(n + 1)$ is bounded because $0 < a_n < 1$ for all n .

Sequences

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leq a_n \leq 1$ but is divergent] and not every monotonic sequence is convergent ($a_n = n \rightarrow \infty$).

But if a sequence is both bounded *and* monotonic, then it must be convergent.

Sequences

This fact is stated without proof as Theorem 12, but intuitively you can understand why it is true by looking at Figure 12.

If $\{a_n\}$ is increasing and $a_n \leq M$ for all n , then the terms are forced to crowd together and approach some number L .

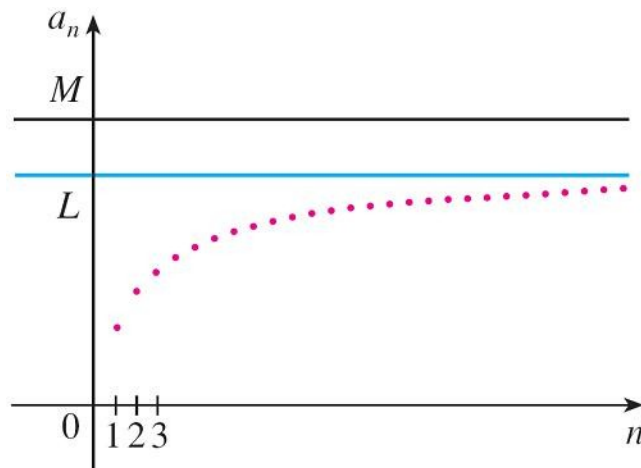


Figure 12

Sequences

The proof of Theorem 12 is based on the **Completeness Axiom** for the set of \mathbb{R} real numbers, which says that if S is a nonempty set of real numbers that has an upper bound M ($x \leq M$ for all x in S), then S has a **least upper bound** b .

(This means that b is an upper bound for S , but if M is any other upper bound, then $b \leq M$.)

Sequences

The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

12 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.