

# 11

# Infinite Sequences and Series



# 11.4

# The Comparison Tests

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# The Comparison Tests

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. For instance, the series

$$\boxed{1} \quad \sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

reminds us of the series  $\sum_{n=1}^{\infty} 1/2^n$ , which is a geometric series with  $a = \frac{1}{2}$  and  $r = \frac{1}{2}$  and is therefore convergent. Because the series  $\boxed{1}$  is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is.

# The Comparison Tests

The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that our given series  $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$  has smaller terms than those of the geometric series and therefore all its partial sums are also smaller than 1 (the sum of the geometric series).

This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

# The Comparison Tests

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent.

The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

**The Comparison Test** Suppose that  $\sum a_n$  and  $\sum b_n$  are series with positive terms.

- (i) If  $\sum b_n$  is convergent and  $a_n \leq b_n$  for all  $n$ , then  $\sum a_n$  is also convergent.
- (ii) If  $\sum b_n$  is divergent and  $a_n \geq b_n$  for all  $n$ , then  $\sum a_n$  is also divergent.

# The Comparison Tests

In using the Comparison Test we must, of course, have some known series  $\sum b_n$  for the purpose of comparison. Most of the time we use one of these series:

- A  $p$ -series [ $\sum 1/n^p$  converges if  $p > 1$  and diverges if  $p \leq 1$ ]
- A geometric series [ $\sum ar^{n-1}$  converges if  $|r| < 1$  and diverges if  $|r| \geq 1$ ]

# Example 1

Determine whether the series  $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$  converges or diverges.

## Solution:

For large  $n$  the dominant term in the denominator is  $2n^2$ , so we compare the given series with the series  $\sum 5/(2n^2)$ .

Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator.

(In the notation of the Comparison Test,  $a_n$  is the left side and  $b_n$  is the right side.)

# Example 1 – Solution

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We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a  $p$ -series with  $p = 2 > 1$ .

Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part (i) of the Comparison Test.

# The Comparison Tests

## **Note 1:**

Although the condition  $a_n \leq b_n$  or  $a_n \geq b_n$  in the Comparison Test is given for all  $n$ , we need verify only that it holds for  $n \geq N$ , where  $N$  is some fixed integer, because the convergence of a series is not affected by a finite number of terms.

# The Comparison Tests

## **Note 2:**

The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series.

If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn't apply.

# The Comparison Tests

Consider, for instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

is useless as far as the Comparison Test is concerned because  $\sum b_n = \sum \left(\frac{1}{2}\right)^n$  is convergent and  $a_n > b_n$ .

# The Comparison Tests

Nonetheless, we have the feeling that  $\sum 1/(2^n - 1)$  ought to be convergent because it is very similar to the convergent geometric series  $\sum \left(\frac{1}{2}\right)^n$ .

In such cases the following test can be used.

**The Limit Comparison Test** Suppose that  $\sum a_n$  and  $\sum b_n$  are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where  $c$  is a finite number and  $c > 0$ , then either both series converge or both diverge.

# Example 3

Test the series  $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$  for convergence or divergence.

**Solution:**

We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \quad b_n = \frac{1}{2^n}$$

and obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{1/(2^n - 1)}{1/2^n} \\ &= \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} \end{aligned}$$

# Example 3 – *Solution*

cont'd

$$= \lim_{n \rightarrow \infty} \frac{1}{1 - 1/2^n}$$

$$= 1 > 0$$

Since this limit exists and  $\sum 1/2^n$  is a convergent geometric series, the given series converges by the Limit Comparison Test.



# Estimating Sums

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If we have used the Comparison Test to show that a series  $\sum a_n$  converges by comparison with a series  $\sum b_n$ , then we may be able to estimate the sum  $\sum a_n$  by comparing remainders.

We consider the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

# Estimating Sums

For the comparison series  $\sum b_n$  we consider the corresponding remainder

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$$

Since  $a_n \leq b_n$  for all  $n$ , we have  $R_n \leq T_n$ . If  $\sum b_n$  is a  $p$ -series, we can estimate its remainder  $T_n$ . If  $\sum b_n$  is a geometric series, then  $T_n$  is the sum of a geometric series and we can sum it exactly.

In either case we know that  $R_n$  is smaller than  $T_n$ .

# Example 5

Use the sum of the first 100 terms to approximate the sum of the series  $\sum 1/(n^3 + 1)$ . Estimate the error involved in this approximation.

**Solution:**

Since

$$\frac{1}{n^3 + 1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test.

# Example 5 – Solution

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The remainder  $T_n$  for the comparison series  $\sum 1/n^3$  was estimated earlier using the Remainder Estimate for the Integral Test.

There we found that

$$T_n \leq \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}$$

Therefore the remainder  $R_n$  for the given series satisfies

$$R_n \leq T_n \leq \frac{1}{2n^2}$$

# Example 5 – Solution

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With  $n = 100$  we have

$$R_{100} \leq \frac{1}{2(100)^2} = 0.00005$$

Using a programmable calculator or a computer, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \approx \sum_{n=1}^{100} \frac{1}{n^3 + 1} \approx 0.6864538$$

with error less than 0.00005.