

11

Infinite Sequences and Series



11.3

The Integral Test and Estimates of Sums

The Integral Test and Estimates of Sums

In general, it is difficult to find the exact sum of a series.

We were able to accomplish this for geometric series and the series $\sum 1/[n(n + 1)]$ because in each of those cases we could find a simple formula for the n th partial sum s_n .

But usually it isn't easy to discover such a formula.

The Integral Test and Estimates of Sums

We begin by investigating the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$

There's no simple formula for the sum s_n of the first n terms, but the computer-generated table of values given to the right suggests that the partial sums are approaching a number near 1.64 as $n \rightarrow \infty$ and so it looks as if the series is convergent.

n	$s_n = \sum_{i=1}^n \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

The Integral Test and Estimates of Sums

We can confirm this impression with a geometric argument. Figure 1 shows the curve $y = 1/x^2$ and rectangles that lie below the curve.

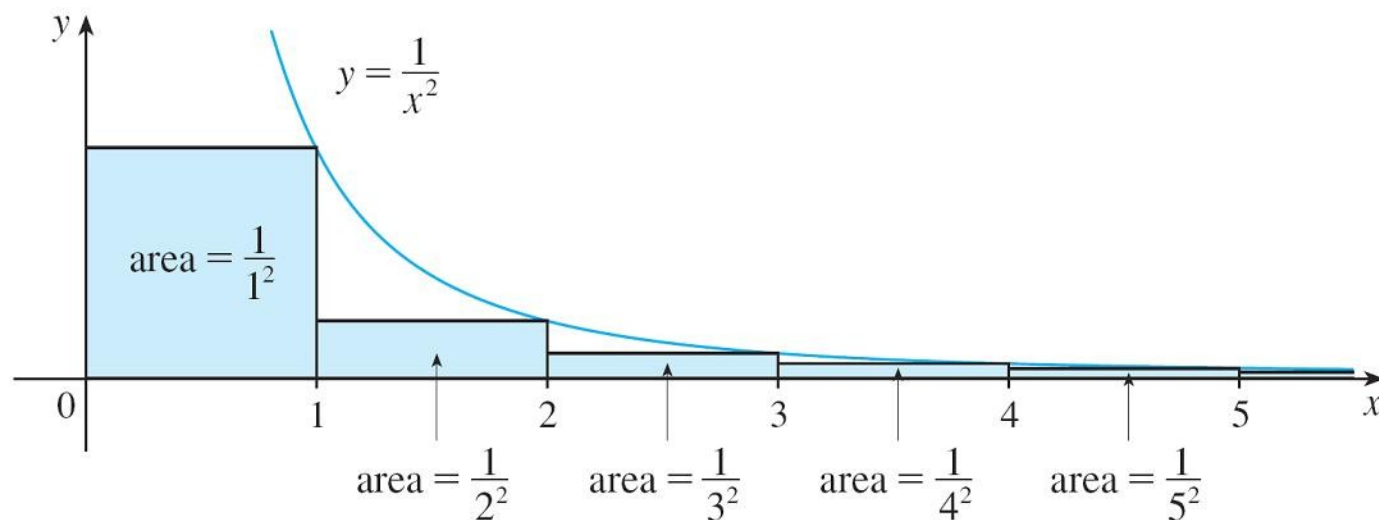


Figure 1

The base of each rectangle is an interval of length 1; the height is equal to the value of the function $y = 1/x^2$ at the right endpoint of the interval.

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So the sum of the areas of the rectangles is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y = 1/x^2$ for $x \geq 1$, which is the value of the integral $\int_1^{\infty} (1/x^2) dx$.

The improper integral is convergent and has value 1. So the picture shows that all the partial sums are less than

$$\frac{1}{1^2} + \int_1^{\infty} \frac{1}{x^2} dx = 2$$

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Thus the partial sums are bounded. We also know that the partial sums are increasing (because all the terms are positive). Therefore the partial sums converge (by the Monotonic Sequence Theorem) and so the series is convergent. The sum of the series (the limit of the partial sums) is also less than 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots < 2$$

Now let's look at the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$$

The Integral Test and Estimates of Sums

The table of values of s_n suggests that the partial sums aren't approaching a finite number, so we suspect that the given series may be divergent.

n	$s_n = \sum_{i=1}^n \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

The Integral Test and Estimates of Sums

Again we use a picture for confirmation. Figure 2 shows the curve $y = 1/\sqrt{x}$, but this time we use rectangles whose tops lie above the curve.

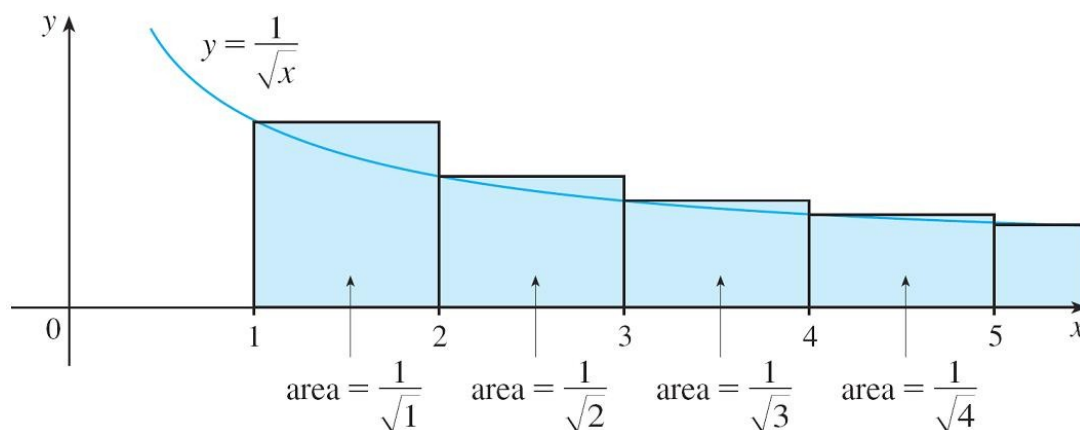


Figure 2

The base of each rectangle is an interval of length 1. The height is equal to the value of the function $y = 1/\sqrt{x}$ at the *left* endpoint of the interval.

The Integral Test and Estimates of Sums

So the sum of the areas of all the rectangles is

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This total area is greater than the area under the curve $y = 1/\sqrt{x}$, for $x \geq 1$, which is equal to the integral $\int_1^{\infty} (1/\sqrt{x}) dx$.

But we know that this improper integral is divergent. In other words, the area under the curve is infinite. So the sum of the series must be infinite, that is, the series is divergent.

The Integral Test and Estimates of Sums

The same sort of geometric reasoning that we used for these two series can be used to prove the following test.

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x) dx$ is convergent. In other words:

(a) If $\int_1^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\int_1^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 1

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ for convergence or divergence.

Solution:

The function $f(x) = 1/(x^2 + 1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2 + 1} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \tan^{-1} x \Big|_1^t \\ &= \lim_{t \rightarrow \infty} \left(\tan^{-1} t - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

Thus $\int_1^{\infty} 1/(x^2 + 1) dx$ is a convergent integral and so, by the Integral Test, the series $\sum 1/(n^2 + 1)$ is convergent.

The Integral Test and Estimates of Sums

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the ***p*-series**.

1 The *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example 3

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots$$

is convergent because it is a p -series with $p = 3 > 1$.

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \dots$$

is divergent because it is a p -series with $p = \frac{1}{3} < 1$.

Example 4

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution:

The function $f(x) = (\ln x)/x$ is positive and continuous for $x > 1$ because the logarithm function is continuous.

But it is not obvious whether or not f is decreasing, so we compute its derivative:

$$\begin{aligned} f'(x) &= \frac{x(1/x) - \ln x}{x^2} \\ &= \frac{1 - \ln x}{x^2} \end{aligned}$$

Example 4

Thus $f'(x) < 0$ when $\ln x > 1$, that is, $x > e$. It follows that f is decreasing when $x > e$ and so we can apply the Integral Test:

$$\begin{aligned}\int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \left. \frac{(\ln x)^2}{2} \right]_1^t \\ &= \lim_{t \rightarrow \infty} \frac{(\ln t)^2}{2} \\ &= \infty\end{aligned}$$

Since this improper integral is divergent, the series $\sum (\ln n)/n$ is also divergent by the Integral Test.



Estimating the Sum of a Series

Estimating the Sum of a Series

Suppose we have been able to use the Integral Test to show that a series $\sum a_n$ is convergent and we now want to find an approximation to the sum s of the series.

Of course, any partial sum s_n is an approximation to s because $\lim_{n \rightarrow \infty} s_n = s$. But how good is such an approximation? To find out, we need to estimate the size of the **remainder**

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder R_n is the error made when s_n , the sum of the first n terms, is used as an approximation to the total sum.

Estimating the Sum of a Series

We use the same notation and ideas as in the Integral Test, assuming that f is decreasing on $[n, \infty)$. Comparing the areas of the rectangles with the area under $y = f(x)$ for $x > n$ in Figure 3, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^{\infty} f(x) dx$$

Similarly, we see from Figure 4 that

$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^{\infty} f(x) dx$$

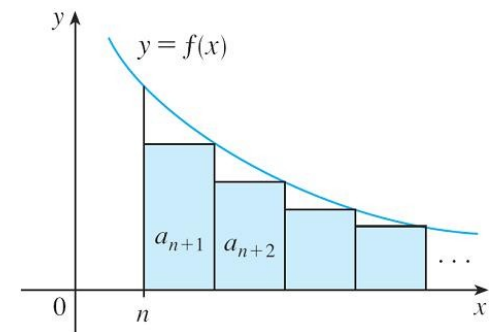


Figure 3

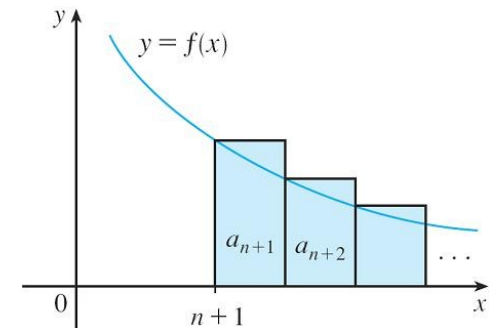


Figure 4

Estimating the Sum of a Series

So we have proved the following error estimate.

2 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

Example 5

- (a)** Approximate the sum of the series $\Sigma 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.
- (b)** How many terms are required to ensure that the sum is accurate to within 0.0005?

Solution:

In both parts (a) and (b) we need to know $\int_n^\infty f(x) dx$. With $f(x) = 1/x^3$, which satisfies the conditions of the Integral Test, we have

$$\int_n^\infty \frac{1}{x^3} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{2x^2} \right]_n^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{2t^2} + \frac{1}{2n^2} \right) = \frac{1}{2n^2}$$

Example 5 – Solution

cont'd

(a) Approximating the sum of the series by the 10th partial sum, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^3} &\approx s_{10} \\ &= \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \\ &\approx 1.1975\end{aligned}$$

According to the remainder estimate $\boxed{2}$, we have

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3} dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

Example 5 – Solution

cont'd

(b) Accuracy to within 0.0005 means that we have to find a value of n such that $R_n \leq 0.0005$. Since

$$R_n \leq \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}$$

we want

$$\frac{1}{2n^2} < 0.0005$$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000 \quad \text{or} \quad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005.

Estimating the Sum of a Series

If we add s_n to each side of the inequalities in [2], we get

3

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq s \leq s_n + \int_n^{\infty} f(x) dx$$

because $s_n + R_n = s$. The inequalities in [3] give a lower bound and an upper bound for s .

They provide a more accurate approximation to the sum of the series than the partial sum s_n does.