

11

Infinite Sequences and Series



11.5

Alternating Series

Alternating Series

In this section we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series*, whose terms alternate in sign.

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \dots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

Alternating Series

We see from these examples that the n th term of an alternating series is of the form

$$a_n = (-1)^{n-1}b_n \quad \text{or} \quad a_n = (-1)^nb_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Alternating Series

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad b_n > 0$$

satisfies

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n \rightarrow \infty} b_n = 0$

then the series is convergent.

Example 1

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

(i) $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$

(ii) $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

so the series is convergent by the Alternating Series Test. 6



Estimating Sums

Estimating Sums

A partial sum s_n of any convergent series can be used as an approximation to the total sum s , but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$.

The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than b_{n+1} , which is the absolute value of the first neglected term.

Estimating Sums

Alternating Series Estimation Theorem If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$(i) \ b_{n+1} \leq b_n \quad \text{and} \quad (ii) \ \lim_{n \rightarrow \infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Example 4

Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

Solution:

We first observe that the series is convergent by the Alternating Series Test because

$$(i) \quad \frac{1}{(n+1)!} = \frac{1}{n!(n+1)} < \frac{1}{n!}$$

$$(ii) \quad 0 < \frac{1}{n!} < \frac{1}{n} \rightarrow 0 \quad \text{so} \quad \frac{1}{n!} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

Example 4 – *Solution*

cont'd

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$\begin{aligned}s &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \dots \\ &= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \dots\end{aligned}$$

Example 4 – *Solution*

cont'd

Notice that

$$b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$$

and

$$\begin{aligned} s_6 &= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \\ &\approx 0.368056 \end{aligned}$$

Example 4 – *Solution*

cont'd

By the Alternating Series Estimation Theorem we know that

$$|s - s_6| \leq b_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have $s \approx 0.368$ correct to three decimal places.

Estimating Sums

Note:

The rule that the error (in using s_n to approximate s) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. **The rule does not apply to other types of series.**