

# 11

# Infinite Sequences and Series



# 11.6

## Absolute Convergence and the Ratio and Root Tests

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# Absolute Convergence and the Ratio and Root Tests

Given any series  $\sum a_n$ , we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

**1 Definition** A series  $\sum a_n$  is called **absolutely convergent** if the series of absolute values  $\sum |a_n|$  is convergent.

# Absolute Convergence and the Ratio and Root Tests

Notice that if  $\sum a_n$  is a series with positive terms, then  $|a_n| = a_n$  and so absolute convergence is the same as convergence in this case.

# Example 1

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

is a convergent  $p$ -series ( $p = 2$ ).

# Example 2

We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is convergent, but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

which is the harmonic series ( $p$ -series with  $p = 1$ ) and is therefore divergent.

# Absolute Convergence and the Ratio and Root Tests

**2 Definition** A series  $\sum a_n$  is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 2 shows that the alternating harmonic series is conditionally convergent. Thus it is possible for a series to be convergent but not absolutely convergent.

However, the next theorem shows that absolute convergence implies convergence.

**3 Theorem** If a series  $\sum a_n$  is absolutely convergent, then it is convergent.

# Example 3

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \dots$$

is convergent or divergent.

## Solution:

This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive: The signs change irregularly.)

# Example 3 – Solution

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We can apply the Comparison Test to the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Since  $|\cos n| \leq 1$  for all  $n$ , we have

$$\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

We know that  $\sum 1/n^2$  is convergent ( $p$ -series with  $p = 2$ ) and therefore  $\sum |\cos n|/n^2$  is convergent by the Comparison Test.

## Example 3 – *Solution*

cont'd

Thus the given series  $\sum (\cos n)/n^2$  is absolutely convergent and therefore convergent by Theorem 3.

# Absolute Convergence and the Ratio and Root Tests

The following test is very useful in determining whether a given series is absolutely convergent.

## The Ratio Test

(i) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent (and therefore convergent).

(ii) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$  or  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.

(iii) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ , the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of  $\sum a_n$ .

# Absolute Convergence and the Ratio and Root Tests

## Note:

Part (iii) of the Ratio Test says that if  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = 1$ , the test gives no information. For instance, for the convergent series  $\sum 1/n^2$  we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1 + \frac{1}{n}\right)^2} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

# Absolute Convergence and the Ratio and Root Tests

whereas for the divergent series  $\sum 1/n$  we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

Therefore, if  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = 1$ , the series  $\sum a_n$  might converge or it might diverge. In this case the Ratio Test fails and we must use some other test.

# Example 5

Test the convergence of the series  $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ .

**Solution:**

Since the terms  $a_n = n^n/n!$  are positive, we don't need the absolute value signs.

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \\ &= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \end{aligned}$$

# Example 5 – Solution

cont'd

as  $n$

$$= \left( \frac{n+1}{n} \right)^n = \left( 1 + \frac{1}{n} \right)^n \rightarrow e \quad \rightarrow \infty$$

Since  $e > 1$ , the given series is divergent by the Ratio Test.

# Absolute Convergence and the Ratio and Root Tests

## Note:

Although the Ratio Test works in Example 5, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \geq n$$

it follows that  $a_n$  does not approach 0 as  $n \rightarrow \infty$ . Therefore the given series is divergent by the Test for Divergence.

# Absolute Convergence and the Ratio and Root Tests

The following test is convenient to apply when  $n$ th powers occur.

## The Root Test

- (i) If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent (and therefore convergent).
- (ii) If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$  or  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.
- (iii) If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ , the Root Test is inconclusive.

# Absolute Convergence and the Ratio and Root Tests

If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ , then part (iii) of the Root Test says that the test gives no information. The series  $\sum a_n$  could converge or diverge.

(If  $L = 1$  in the Ratio Test, don't try the Root Test because  $L$  will again be 1. And if  $L = 1$  in the Root Test, don't try the Ratio Test because it will fail too.)

# Example 6

Test the convergence of the series  $\sum_{n=1}^{\infty} \left( \frac{2n+3}{3n+2} \right)^n$ .

Solution:

$$a_n = \left( \frac{2n+3}{3n+2} \right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2 + \frac{3}{n}}{3 + \frac{2}{n}} \rightarrow \frac{2}{3} < 1$$

Thus the given series converges by the Root Test.



# Rearrangements

# Rearrangements

The question of whether a given convergent series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave like finite sums.

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series.

# Rearrangements

By a **rearrangement** of an infinite series  $\sum a_n$  we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of  $\sum a_n$  could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \cdots$$

It turns out that

if  $\sum a_n$  is absolutely convergent series with sum  $s$ , then any rearrangement of  $\sum a_n$  has the same sum  $s$ .

# Rearrangements

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact let's consider the alternating harmonic series

$$\boxed{6} \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2$$

If we multiply this series by  $\frac{1}{2}$ , we get

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{1}{2} \ln 2$$

# Rearrangements

Inserting zeros between the terms of this series, we have

$$\boxed{7} \quad 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{1}{2} \ln 2$$

Now we add the series in Equations 6 and 7:

$$\boxed{8} \quad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2} \ln 2$$

# Rearrangements

Notice that the series in [8] contains the same terms as in [6], but rearranged so that one negative term occurs after each pair of positive terms. The sums of these series, however, are different. In fact, Riemann proved that

if  $\sum a_n$  is a conditionally convergent series and  $r$  is any real number whatsoever, then there is a rearrangement of  $\sum a_n$  that has a sum equal to  $r$ .